1960

The analysis of third-order servo with backlash and load inertia

Sjamsoeddin, Abdoellah; Tanaka, Katsuji

Monterey, California: U.S. Naval Postgraduate School

THE ANALYSIS OF THIRD-ORDER SERVO
WITH BACKLASH AND LOAD INERTIA

ABDOELLAH SJAMSOEDDIN
and
KATSUJI TANAKA

[illegible]

UNITED STATES
NAVAL POSTGRADUATE SCHOOL

THESIS

THE ANALYSIS OF THIRD-ORDER SERVO WITH

BACKLASH AND LOAD INERTIA

Abdoellah Sjamsoeddin

and

Katsuji Tanaka

THE ANALYSIS OF THIRD-ORDER SERVO WITH

BACKLASH AND LOAD INERTIA

* * * * *

Abdoellah Sjamsoeddin

and

Katsuji Tanaka

THE ANALYSIS OF THIRD-ORDER SERVO WITH BACKLASH AND LOAD INERTIA

by

Abdoellah Sjamsoeddin

Captain, Indonesian Navy

and

Katsuji Tanaka

Lieutenant (j.g.), Japanese Navy

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

United States Naval Postgraduate School

Monterey, California

1960

THE ANALYSIS OF THIRD-ORDER SERVO WITH

BACKLASH AND LOAD INERTIA

by

Abdoellah Sjamsoeddin

and

Katsuji Tanaka

This work is accepted as fulfilling the

thesis requirements for the degree of

MASTER OF SCIENCE

IN

ELECTRICAL ENGINEERING

From the

United States Naval Postgraduate School

PREFACE

Backlash becomes an engineering problem. The backlash ordinarily arises in the gear train connecting the output motor to the load. The performance changes resulting from the addition of backlash depend upon whether the major portion of the output inertia is on the motor or the load side of the backlash. No matter where the servomechanism error is measured, backlash contributes either to the steady state error, or instability of the system.

Backlash effects are investigated by analog computer techniques in this thesis.

We wish to acknowledge the assistance, the comments and suggestions we have received from Professor Dr. George J. Thaler and that make us possible to complete this thesis.

TABLE OF CONTENTS

Section		Page
1.	Background of the problem.	1
2.	Review of second order system with backlash and load inertia.	14
3.	Physical analysis of third order system with backlash and load inertia.	27
4.	Analog computer technique.	65
5.	The results of investigation.	75
6.	Discussion and conclusions.	78

List of Illustrations

			Page
1.	Fig. 1	Block diagram of a simple feedback control system	3
2.	Fig. 2	Relationship between angular position of two gears	3
3.	Fig. 3	A second order servo with backlash	5
4.	Fig. 4	Phase plane trajectory of a second order servo	6
5.	Fig. 5	A sketch of displacement vs. time	7
6.	Fig. 6	A sketch of velocity vs. displacement	10
7.	Fig. 7	A sketch of acceleration vs. displacement	11
8.	Fig. 8	A sketch of torque vs. displacement	12
9.	Fig. 9	A sketch of displacement vs. displacement	13
10.	Table 1		18
11.	Table 2		19
12.	Table 3		20
13.	Table 4		23
14.	Fig. 10	A graph of displacement vs. time	25
15.	Fig. 11	A phase plane trajectory of velocity vs. displacement.	26
16.	Fig. 12	A block diagram of a third order servo	29
17.	Fig. 13	A phase plane trajectory vs. displacement	33
18.	Fig. 14	A plot of acceleration vs. displacement	35
19.	Fig. 15	A plot of motor velocity vs. displacement	41
20.	Fig. 16	A plot of load displacement vs. motor displacement	42
21.	Fig. 17	A plot of motor torque vs. motor displacement	43
22.	Fig. 18	The root locus of the system	44
23.	Table 5		48
24.	Table 6		52

List of illustrations (Cont'd)

Page

25. Table 7 53

26. Fig. 19 A complete phase plane trajectory with backlash .10 rad. 54

27. Fig. 20 A complete phase plane trajectory with backlash .20 rad. 55

28. Fig. 21 A complete phase plane trajectory with backlash .30 rad. 56

29. Fig. 22 A complete phase plane trajectory with backlash .40 rad. 57

30. Fig. 23 A complete phase plane trajectory with backlash .50 rad. 58

31. Fig. 24 A complete phase plane trajectory with backlash .60 rad. 59

32. Fig. 25 A complete phase plane trajectory with backlash .70 rad. 60

33. Fig. 26 A complete phase plane trajectory with backlash .80 rad. 61

34. Fig. 27 A complete phase plane trajectory with backlash .90 rad. 62

35. Fig. 28 A complete phase plane trajectory with backlash 1.0 rad. 63

36. Fig. 29 Trasient response of motor and load after separation. 64

37. Fig. 30 A circuit diagram of computer set up for the combined system. 67

38. Fig. 31 A circuit diagram of computer set up for the load and motor after separation 72

39. Table 8 76

40. Fig. 32 The plot of amplitude of limit cycle vs. backlash with several value of gain 77

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

1. Background of problem.

In the study of backlash we have to consider two cases:

a) Simple backlash theory, in which there is no inertia on load side.

b) Complicated backlash i.e. inertia and friction on load side.

a. Simple backlash theory.

A simple feedback control system is assumed as shown in Fig. 1. Backlash occurs in gear train connecting the output motor to the load. The summer is assumed ideal here in its function of sending to the controller the difference between input signal and output of box containing gears with backlash. The input to the system is a shaft rotation, designated θ_R in Fig. 1. The output shaft is a rotation and designated with θ_C. The output of the gear box, whose output is θ_C, is assumed ideal in all respect except for the presence of backlash. Because there is no inertia on load side the backlash is such a character as shown in Fig. 2. This figure shows that as the output shaft moves with finite velocity from a position θ_C is zero to a position θ_C has some positive value, say θ_{C1} greater than 1/2 the backlash angle, the output of the gear box θ_C' moves from zero to a certain positive value; then the direction of rotation reverses so that θ_C decreases through zero to some negative value θ_{C2} greater than 1/2 the backlash angle. The value of θ_C' goes from positive through zero to negative value and then goes back to zero again. $\triangle$ is the total backlash angle. There is a dead space and this means that the adjacent gear teeth must move through a finite distance before making contact with each other. As long as the operation is unidirection there is no trouble, but soon when it changes its direction, then backlash must be taken up each time. In manual

operation it is sometimes noisy and can cause instability in the system.

b. Qualitative effect of inertia on load.

Assumed we have a unity feedback system as shown in Fig. 1. The discussion of the physical nature of the system is as follows:

In normal operation the motor may be driving in one direction and backlash is then taken up, so that all gears are in contact. If the load inertia is small and there is some load friction, the load shaft will stop almost immediately with the reversal of the motor rotation. The load shaft will remain stationary until the motor has moved through an angle sufficient to take up all backlash. On the other hand, if the load inertia is big enough, and the load friction is small, when the motor reverses the load will not reverse, but may continue to rotate in its original direction with a constant velocity equal to its velocity at the instant the gear tooth contact was broken. If both the load inertia and friction are appreciable, the load rotates in its original direction but with a constant deceleration. Under these conditions the motor shaft and load shaft both help to take up the backlash.

c. Second order servo with backlash.

The schematical diagram of a secondorder servo is shown in Fig. 3. In the normal arrangement the load is geared to the motor through gear train G_1 and the load shaft is geared to a measuring device through gear train G_2. There may be backlash in either of these gear trains or in both of them. For our purpose we consider the backlash in G_2 only. Consider the sketch of Fig. 4. Assuming a step displacement of magnitude OA and also assuming that the backlash happens to be all taken up in the proper direction, the trajectory starts at A. The trajectory

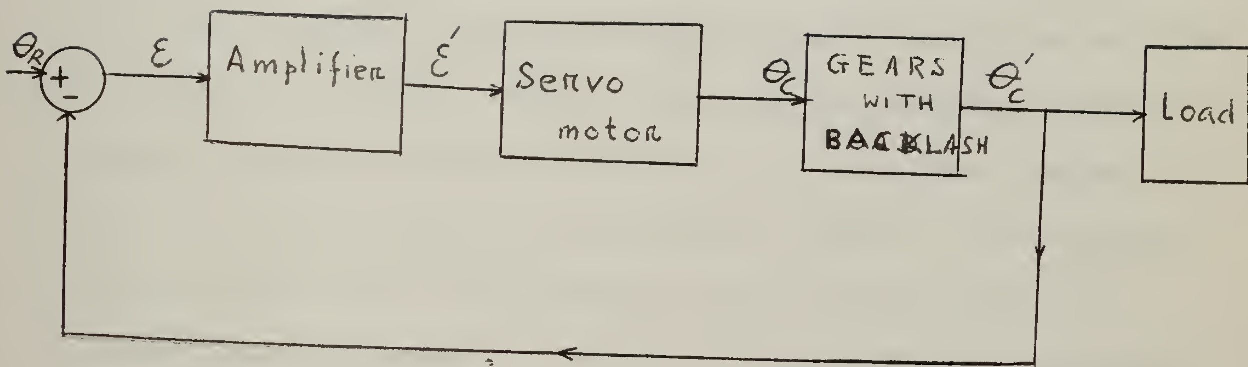

FIG: 1 Block diagram of a simple feedback control system.

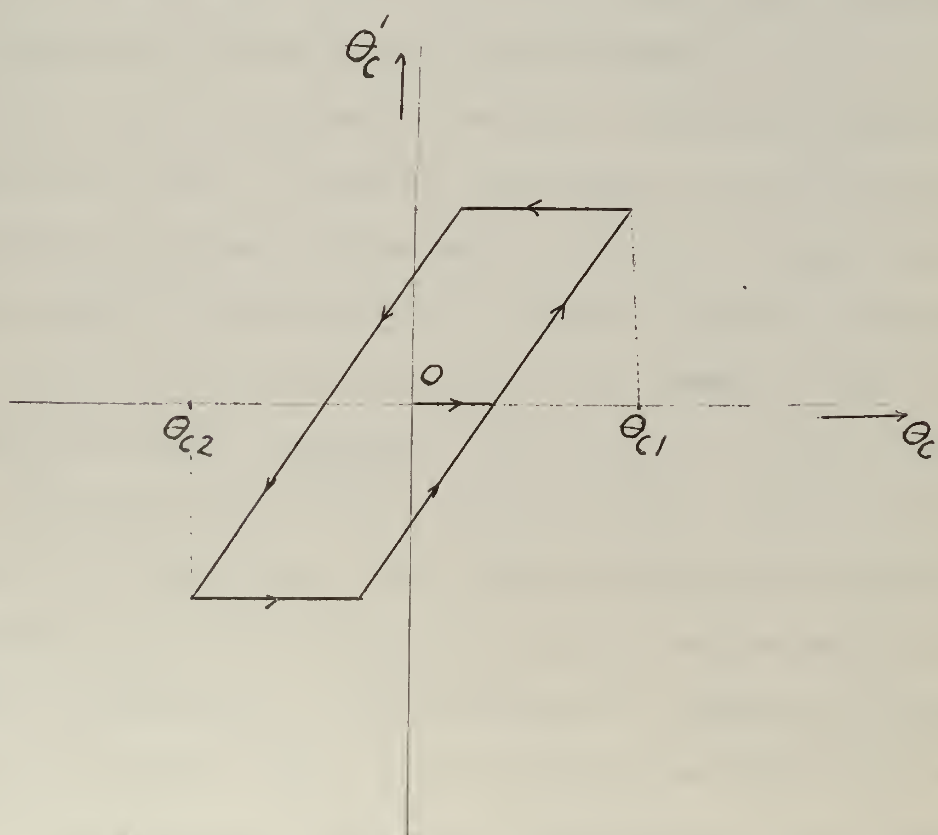

Fig. 2. Relationship between angular position of two gears between which backlash exists.

[illegible]

[illegible]

follows a linear path. At point B the velocities of the motor and load become different, mechanically contact is lost and separation occurs, providing the friction of load has a finite value and the coloumb friction of load can be neglected. The point of separation is determined by the load trajectory, when this trajectory has a common slope with the trajectory of the system. Combining these points together it forms a straight line and is called the "Separation dividing line." After point B the system operates in the backlash region. If the viscous friction of the load is negligible the trajectory of the load is a straight line with constant velocity. But when the friction is sufficient enough the load is drifting separately with a constant deceleration. The remainder of the system continues to drive in the reverse direction until the backlash has taken up and the system output completely corrected as to displacement. The separation dividing line determines the initial conditions for load velocity and displacement, which are the same value as the initial conditions for the motor velocity and displacement.

The sketch of Fig. 5 shows the displacement of load as a function of time and also the motor displacement as a function of time. From these two curves the time at which the backlash is taken up can be determined. Knowing this time the volocity and displacement of load at time of recombination can be determined. This is point C in the sketch of Fig. 4. At the same time point D is also located. This point indicates the velocity and displacement of the system without load. Each pair of corresponding points for load and system without load, velocity and displacement at recombination, in turn determine the velocity and displacement for the recombined system. This is obtained by applying the law of conservation of momentum. The point of recombination for the system is

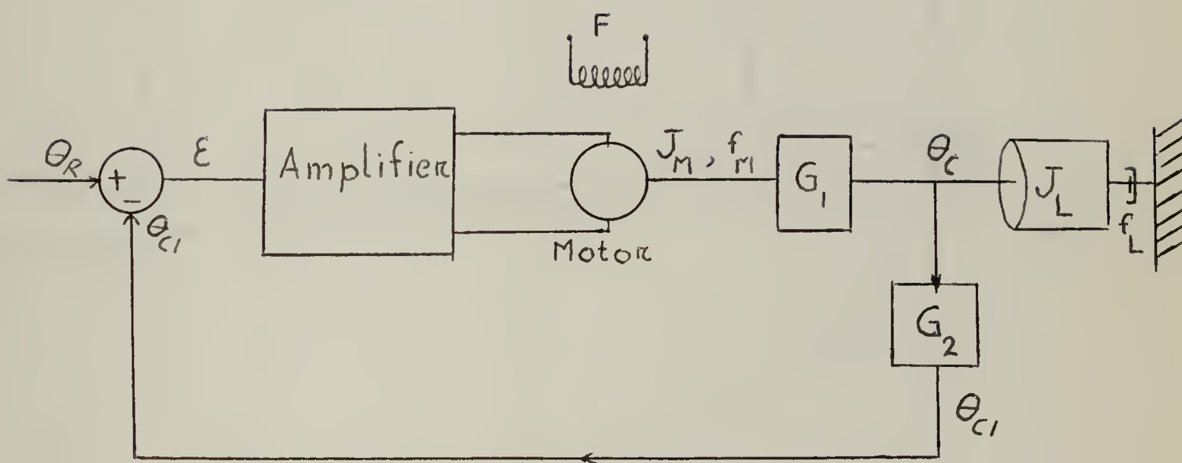

Fig. 3. A second order servo with backlash

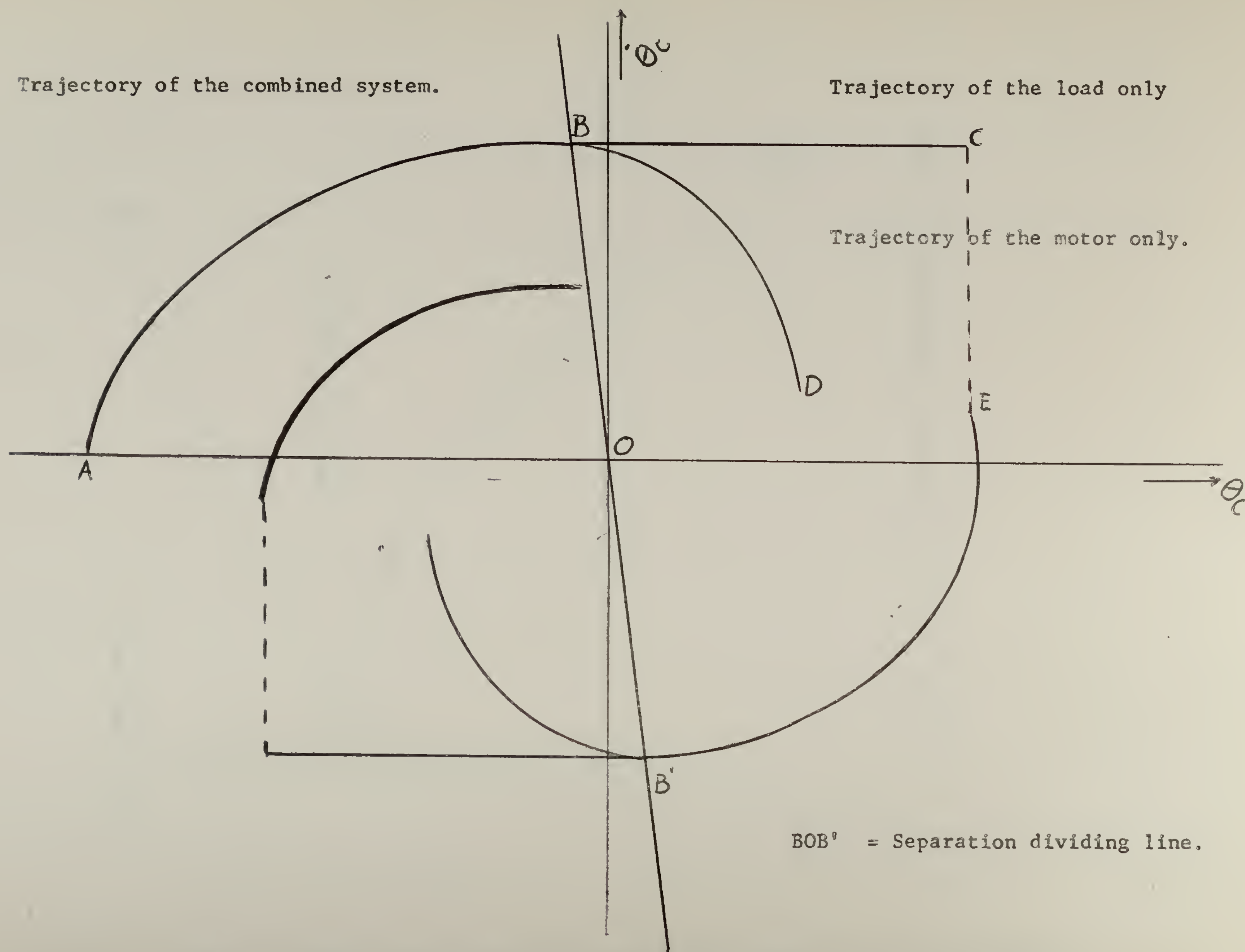

Fig. 4. Phase plane trajectory of a second order servo with backlash.

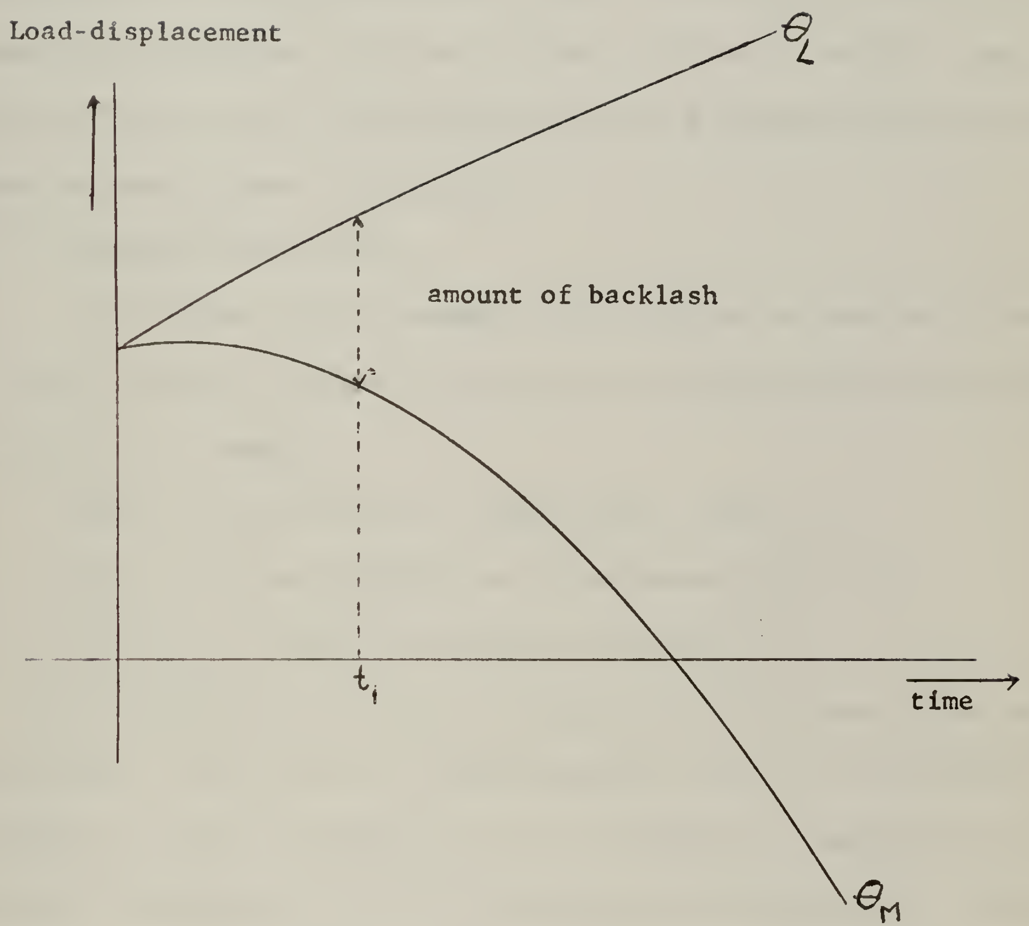

Fig. 5. A sketch of curves of displacement vs time.

point E in Fig. 4. Trajectories for the recombined system start from E and the system retains its original characteristics until the next separation dividing line is reached; separation and recombination will occur again as in the previous half cycle. If the final state of the system spiral into the origin the system is said to be stable. But when the final state of the system oscillates with a constant amplitude, the system is said to have a limit cycle.

d. Higher order system.

For higher order, because of the complex mathematics we have to break it up into several phase plane trajectories in order to analyze the system. For example:

1. The plot of velocity vs. displacement (Fig. 6)
2. The plot of acceleration vs. displacement (Fig. 7)
3. The plot of torque vs. displacement (Fig. 8)

The conditions at the point of separation is the same as the second order system. At the point of recombination, the law of conversation of mementum is still applicable, but we also have to consider the torque equation of the motor, in order to determine the acceleration. Knowing the acceleration and velocity the point of recombination is located. The trajectory of the combined system start again until the separation dividing line is reached. The picture repeats again as in the previous half cycle.

Note here, that for the determination of the load displacement and velocity it is not necessary to go back to the time function. In order to know the value of load displacement and motor displacement at which the backlash is taken up, just plot the load displacement versus motor displacement as shown in Fig. 19. Using this value we go back to Fig. 6, where

the velocity of motor is plotted against the motor displacement. For the calculation of velocity at recombination, the law of conservation of momentum is applicable as the case in the second order system. At the recombination point the torque equation of the motor is true, since the load has no torque at all and also providing the current of the motor is a continuous function of time during the separation. From Fig. 8, we can determine the value of the torque at the time of recombination. Knowing the torque and the velocity, we can evaluate the value of acceleration at this point.

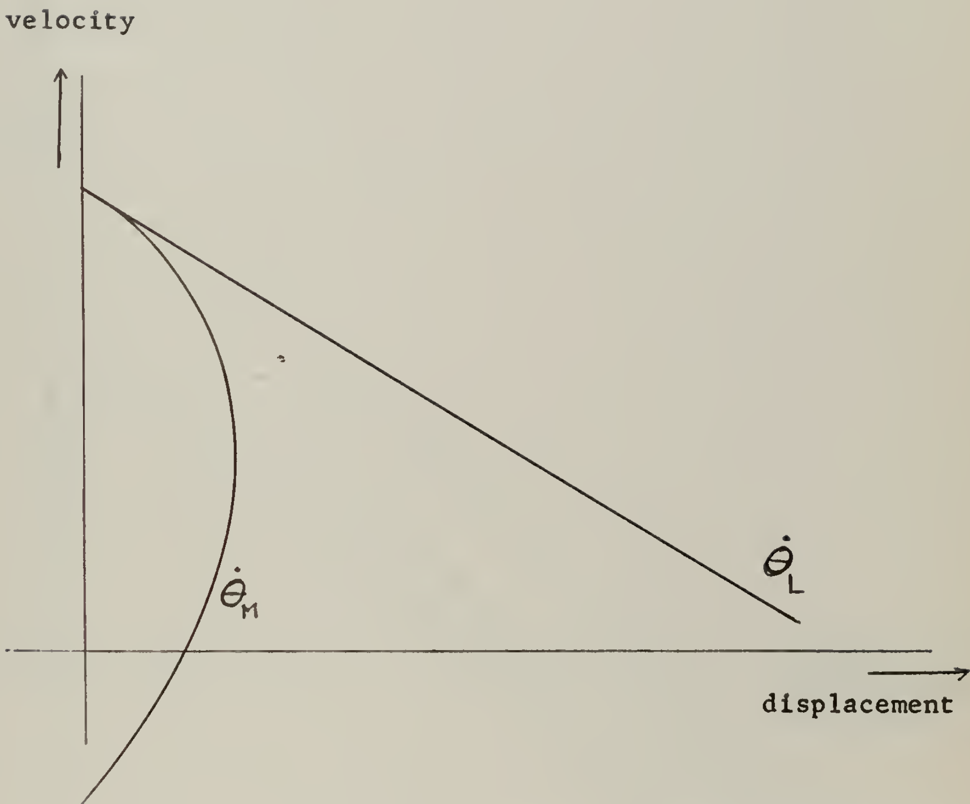

Fig. 6. A sketch of velocity versus displacement.

Fig. 7. A sketch of acceleration vs. displacement of the combined system.

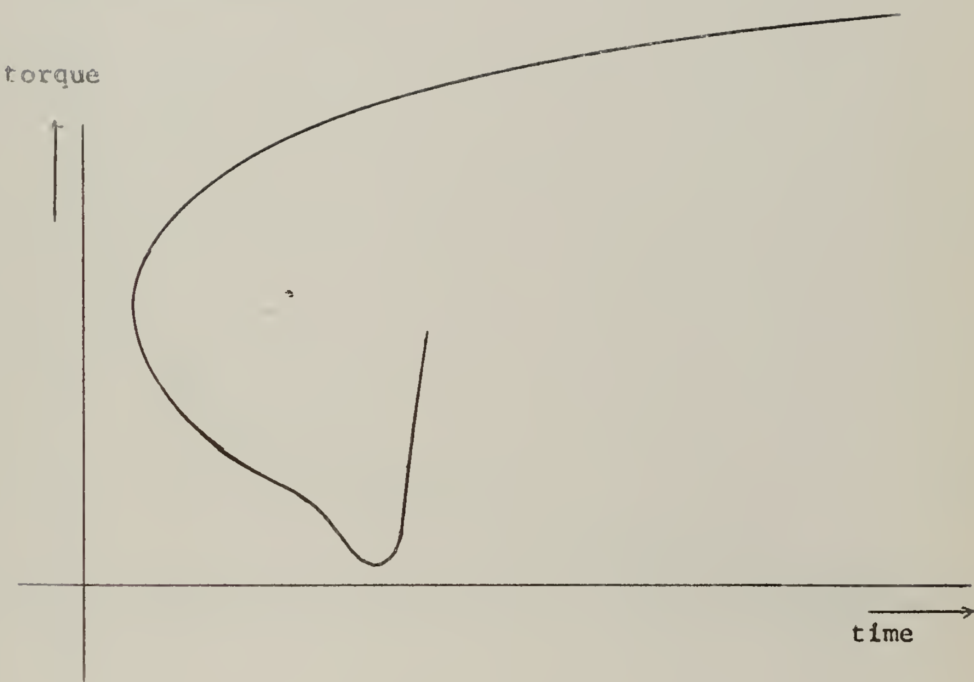

Fig. 8. A sketch of torque vs. displacement of the motor.

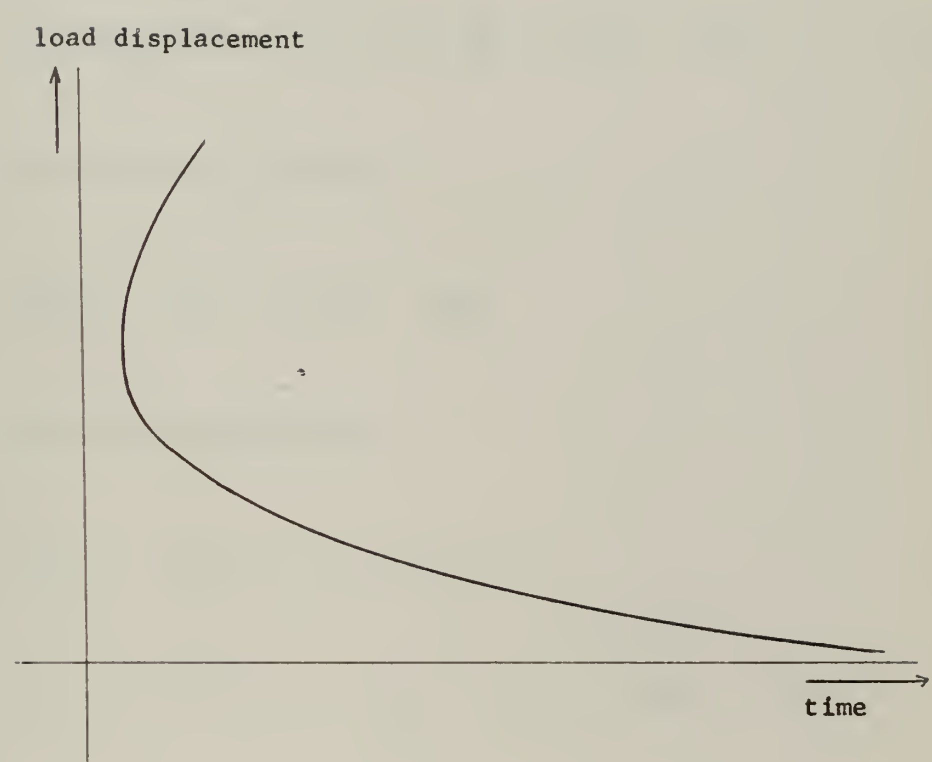

Fig. 9. A sketch of load displacement vs. motor displacement

2. Review of Second Order System with Backlash and Load Inertia.

For our particular problem assume the gear ratio is equal to one, the equation for the combined system becomes:

$$(J_M + J_L)\,\ddot{\theta}_C + (f_M + f_L)\,\dot{\theta}_C + K\theta_C = K\theta_R \qquad (1)$$

For the system with load removed:

$$J_M\ddot{\theta}_M + f_M\dot{\theta}_M + K\theta_M = K\theta_R \qquad (2)$$

For the load drifting separately:

$$J_L\ddot{\theta}_L + f_L\dot{\theta}_L = 0 \qquad (3)$$

Given: $J_M + J_L = 1$ $\quad f_M + f_L = 0.40 \quad K = 1.0$

$J_M = 0.50 \quad f_M = 0.30 \quad \theta_R = 1.0$

$J_L = 0.50 \quad f_L = 0.10 \quad \Delta = 0.30$ rad.

Equation (1) becomes:

$$1\ddot{\theta}_C + 0.40\dot{\theta}_C + \theta_C = 1 \qquad (4)$$

To obtain isoclines for the combined system:

$$\frac{\ddot{\theta}_C}{\dot{\theta}_C} + 0.40 = \frac{1 - \theta_C}{\dot{\theta}_C} \qquad (5)$$

$$\text{Let} \quad N_1 = \frac{\ddot{\theta}_C}{\dot{\theta}_C}$$

$$N_1 + 0.40 = \frac{1 - \theta_C}{\dot{\theta}_C} \tag{5a}$$

Computed values of θ_C and $\dot{\theta}_C$ for various values of N_1 are listed in Table one.

The system as open loop with load separated:

$$0.50\ddot{\theta}_M + 0.30\dot{\theta}_M = 1 - \theta_L \tag{6}$$

At the instant of separation:

$$\frac{\ddot{\theta}_M}{\dot{\theta}_M} + \underline{0.30} = \frac{1 - \theta_{Lo}}{\dot{\theta}_{Mo}} \tag{7}$$

At the instant of separation:

$$\theta_{Co} = \theta_{Mo} = \theta_{Lo}$$

$$\dot{\theta}_{Co} = \dot{\theta}_{Mo} = \dot{\theta}_{Lo}$$

Hence the equation (7) becomes:

$$N_2 + 0.60 = \frac{2(1 - \theta_{Mo})}{\dot{\theta}_{Mo}} \tag{7a}$$

$$\text{where} \quad N_2 = \frac{\ddot{\theta}_M}{\dot{\theta}_M}$$

$$\frac{\dot{\theta}_{Mo}}{1 - \theta_{Mo}} = \frac{2}{N_2 + 0.60} \tag{8}$$

Upon separation of the load, equation (3) becomes:

$$0.50\ddot{\theta}_L + 0.10\dot{\theta}_L = 0 \tag{9}$$

$$N_3 = \frac{\ddot{\theta}_L}{\dot{\theta}_L} = -0.20 \tag{10}$$

Letting $N_2 = N_3 = -0.20$, equation (8) becomes:

$$\frac{\dot{\theta}_{MO}}{1 - \theta_{Mo}} = 5.0$$

hence $\theta_{Mo} = (1 - 0.20)\ \dot{\theta}_{Mo}$ (11)

$$\text{arc. tan.}\ \frac{\dot{\theta}_{Mo}}{1 - \theta_{Mo}} = 78.7°$$

$$\phi = 101.3°$$

Solve for equation)9) as a function of time we obtain:

$$0.50\ (s^2\theta_L - s\theta_{Lo} - \dot{\theta}_{Lo}) + 0.10\ (s\theta_L - \theta_{Lo}) = 0 \tag{12}$$

$$s\ (.5\ s + .1)\ \theta_L = (.5\ s + .1)\theta_{Lo} + .5\ \dot{\theta}_{Lo} \tag{13}$$

where $\dot{\theta}_{Lo} = \dot{\theta}_{Mo} = \dot{\theta}_{Co}$

Equation (13) becomes:

$$\theta_L = \frac{1}{s}\theta_{Mo} + \frac{1}{s\ (s - .2)}\dot{\theta}_{Mo} \tag{14}$$

Substitute the value of equation (11) into equation (14) it becomes:

$$\theta_L = \frac{(1 - .2)}{s} \dot{\theta}_{Mo} + \frac{1}{s(s - .2)} \dot{\theta}_{Mo} \quad (15)$$

hence: $$\theta_L = (1 - .20)\ \dot{\theta}_{Mo} + 5.0\ \dot{\theta}_{Mo}\ (1 - e^{-.20t}) \quad (16)$$

Values of θ_L versus time (t) are tabulated in Table 1.

TABLE 1.

t	$(1 - e^{-.2t})$	$5(1 - e^{-.2t})$	-.2	$\theta_L - 1$
.2	.0392	.196	-.2	.004
.4	.077	.385	-.2	.185
.6	.113	.565	-.2	.365
1.0	.181	.905	-.2	.705
1.4	.244	1.24	-.2	1.04
1.8	.302	1.52	-.2	1.32
2.2	.356	1.78	-.2	1.58
2.6	.405	2.205	-.2	1.825
3.0	.450	2.25	-.2	2.05
3.4	.493	2.465	-.2	2.265
3.8	.532	2.66	-.2	2.46
4.0	.550	2.75	-.2	2.55

Fig. 10 is a graph of equations (16) and (17) . By appropriate ordinate scaling, recombination times were computed as listed in table number four; for corresponding values of $\ddot{\theta}_{Mo}$. Values of θ_M, $\dot{\theta}_M$, θ_L, $\dot{\theta}_L$ for dividing lines are listed in Table four.

Substitute the value of equation (15) into equation (6) and solve for θ_M as a function of time:

$$\theta_M = -16\,\ddot{\theta}_{Mo}t + 111.455\,\ddot{\theta}_{Mo} + 13.345\,\ddot{\theta}_{Mo}e^{-.6t} - 125\,\ddot{\theta}_{Mo}e^{-.2t} + 1.00 \tag{17}$$

Values of θ_M and $\ddot{\theta}_{Mo}$ versus time (t) are tabulated in Table three.

[illegible]

TABLE 2

t	$e^{-.2t}$	$e^{-.65}$	$-125e^{-.2t}$	$13.345\ e^{-.6t}$	-16t	111.455	θ_M -1
.2	.9608	.887	-120	11.82	-3.2	111.455	.075
.4	.923	.787	-115.15	10.5	-6.4	111.455	.105
.6	.887	.698	-110.9	9.305	-9.6	111.455	.260
.8	.8525	.620	-106.5	8.28	-12.8	111.455	.430
1.0	.819	.550	-102.1	7.35	-16	111.455	.70
1.2	.787	.487	- 98.25	6.50	-19.2	111.455	.505
1.4	.756	.432	- 94.5	5.76	-22.4	111.455	.315
1.6	.726	.384	- 90.85	5.12	-25.6	111.455	.125
1.8	.698	.340	- 87.25	4.545	-28.8	111.455	-.05
2.0	.670	.302	- 83.8	4.035	-32.0	111.455	-.310
2.2	.644	.268	- 80.45	3.58	-35.2	111.455	-.615
2.4	.620	.237	- 77.5	3.16	-38.4	111.455	-1.285
2.6	.595	.210	- 74.45	2.805	-41.6	111.455	-1.790
2.8	.572	.187	- 71.5	2.50	-44.8	111.455	-2.345
3.0	.550	.166	- 68.8	2.22	-48.0	111.455	-3.125
3.2	.528	.147	- 66.0	1.961	-51.2	111.455	-3.784
3.4	.507	.130	- 63.4	1.735	-54.4	111.455	-4.61
3.6	.487	.116	- 60.95	1.55	-57.6	111.455	-5.545
3.8	.468	.102	- 58.5	1.361	-60.8	111.455	-6.484
4.0	.450	.091	- 56.2	1.213	-64.0	111.455	-7.532

TABLE 3

t	$25\ e^{-.25}$	$-\ 8\ e^{-.6t}$	$-\ 16$	$\tilde{\theta}_M$
.2	24	-7.1	-16	.90
.4	23.03	-6.30	-16	.73
.6	22.15	-5.575	-16	.573
.8	21.3	-4.96	-16	.34
1.0	20.42	-4.40	-16	.02
1.2	19.65	-3.895	-16	- .245
1.4	18.90	-3.455	-16	- .445
1.6	18.15	-3.065	-16	- .915
1.8	17.41	-2.72	-16	-1.31
2.0	16.73	-2.42	-16	-1.69
2.2	16.07	-2.14	-16	-2.07
2.4	15.49	-1.895	-16	-2.405
2.6	14.86	-1.68	-16	-2.82
2.8	14.30	-1.495	-16	-3.195
3.0	13.73	-1.33	-16	-3.60
3.2	13.19	-1.18	-16	-3.99
3.4	12.66	-1.04	-16	-4.38
3.6	12.16	- .929	-16	-4.769
3.8	11.69	- .816	-16	-5.126
4.0	11.22	- .728	-16	-5.508

TABLE 3

θ_{Mo}	t	$e^{-.2t}$	$e^{-.6t}$	$-.25e^{-.2t}$	$13.345\ e^{-.6t}$	-16t	111.455	$\theta_M - 1$
1.0	1.1	.803	.517	-100.4	6.9	-17.6	111.455	.355
.8	1.175	.791	.494	- 99.0	6.6	-18.8	111.455	.204
.6	1.275	.775	.466	- 96.9	6.225	-20.4	111.455	.228
.4	1.425	.752	.426	- 94.0	5.7	-22.8	111.455	.142
.2	1.875	.688	.3245	- 86.0	4.335	-30.0	111.455	.044
.1	2.425	.616	.234	- 77.0	3.125	-38.8	111.455	-.122
.08	2.64	.590	.206	- 73.8	2.75	-42.25	111.455	-.1475
.06	3.2	.528	.147	- 66.0	1.96	-51.2	111.455	-.227
.04	3.525	.494	.1205	- 61.6	1.61	-56.5	111.455	-.201
.03	3.98	.452	.092	- 56.5	1.23	-63.6	111.455	-.2225

TABLE 3 continued

θ_{Mo}	$25\ e^{-.2t}$	$-\ 8.0\ e^{-.6t}$	$-\ 16$	$\tilde{\theta}_M$
1.0	20.08	-4.135	-16	-.055
.8	19.8	-3.94	-16	-.112
.6	19.35	-3.73	-16	-.222
.4	18.18	-3.41	-16	-.244
.2	17.2	-2.60	-16	-.28
.1	15.4	-1.87	-16	-.247
.08	14.75	-1.65	-16	-.232
.06	13.2	-1.175	-16	-.238
.04	12.32	-.965	-16	-.151
.03	11.3	-.735	-16	-.113

TABLE 3 continued

θ_{Mo}	t	$\ddot{\theta}_L$	θ_L - 1
1.0	1.1	.803	.785
.8	1.175	.632	.675
.6	1.275	.465	.555
.4	1.425	.30	.4155
.2	1.875	.1375	.272
.1	2.425	.0616	.172
.08	2.64	.047	.148
.06	3.2	.0316	.1292
.04	3.525	.01975	.0933
.03	3.98	.01352	.0761

To satisfy the principle of conservation of momentum, applied the next equation:

$$(J_M + J_L) \ \dot{\theta}_C = J_M \dot{\theta}_M + J_L \dot{\theta}_L \tag{18}$$

From equation (18) $\dot{\theta}_M$ and $\dot{\theta}_L$ are known, hence we can solve this equation. The values of θ_C upon recombination corresponding to values of $\dot{\theta}_{Mo}$ upon separation are listed in Table 4

TABLE 4

$\ddot{\theta}_{Mo}$	$\dot{\theta}_C$
1.0	.374
.8	.260
.6	.1216
.4	.029
.2	.07125
.1	-.0927
.08	-.0925
.06	-.1032
.04	-.0656
.03	-.0497

Fig. 11 is the phase plane presentation of θ_C, $\dot{\theta}_C$, θ_L, and $\dot{\theta}_L$ as a combined system and when operating in the backlash region. A step input of 0.80 is seen to separate the load from the system at A, recombination takes place when the load is at B, and the balancing of momentum between load and the system without load, causes θ_C, $\dot{\theta}_C$ of the recombined system to originate from point C. Reseparation occurs at point D and the trajectory is seen to spiral into a limit cycle defined by points E, F, G and H. The magnitude of the limit cycle is 0.550.

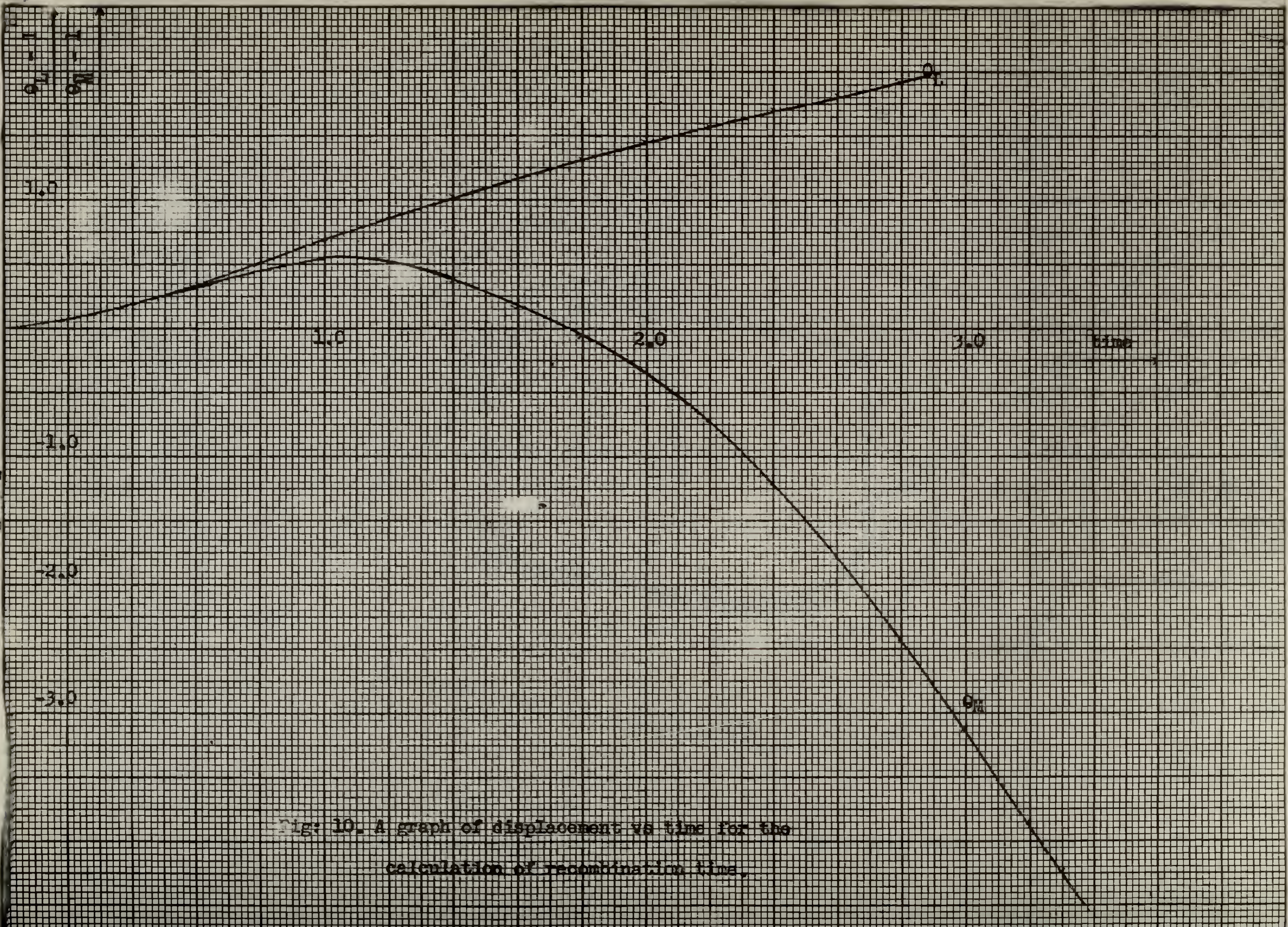

Fig. 10. A graph of displacement vs time for the calculation of recombination time.

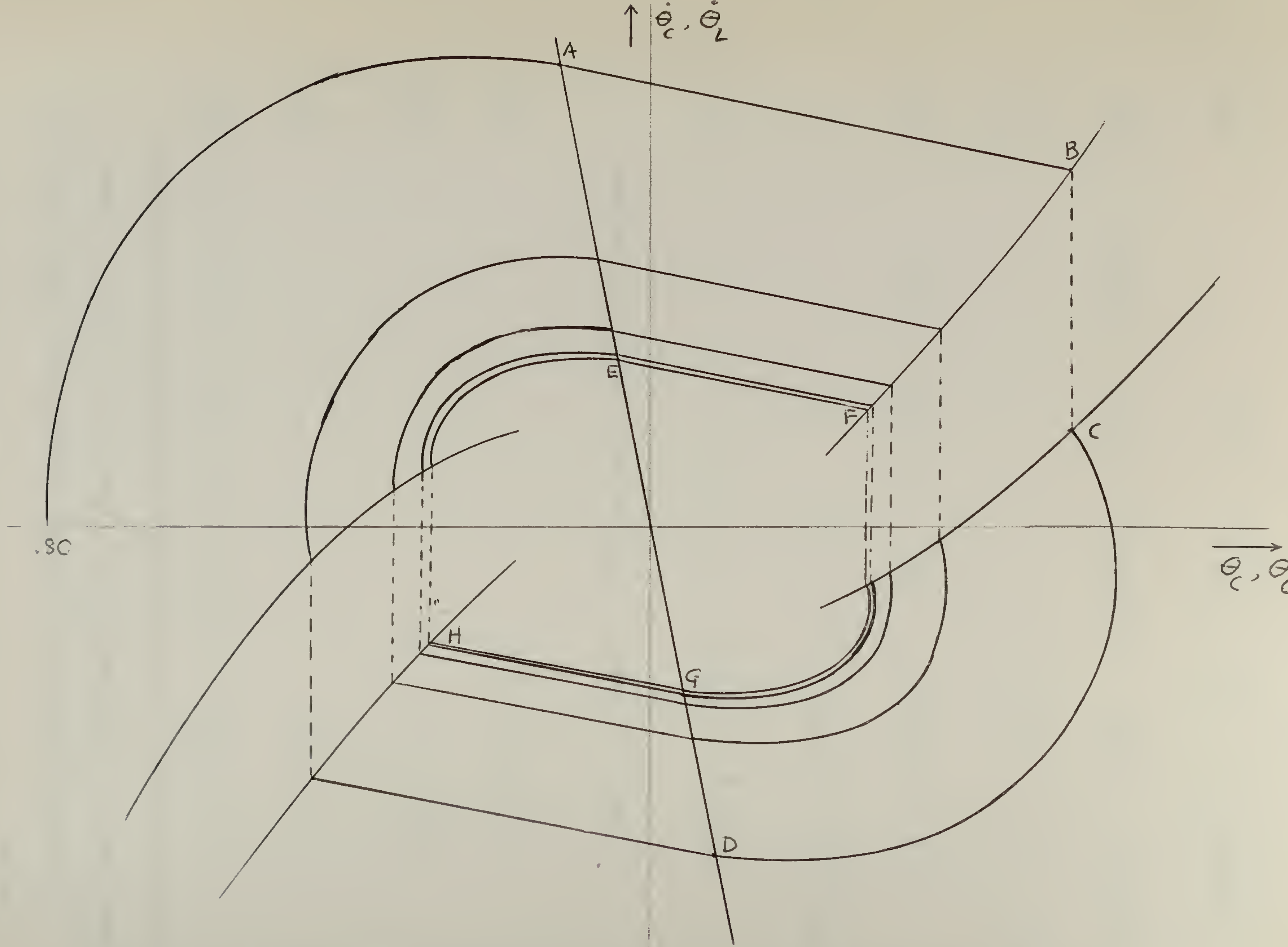

Fig. 11. A phase plane trajectory of 0_C vs. 0_C for a step input of .80.

3. Physical analysis of Third Order System with Backlash and Load Inertia.

The backlash existing in the gear train between motor and load is placed inside the feedback loop as illustrated in Fig. 12. When the backlash is taken up, the differential equations for the system are:

$$1/N \ (N^2 J_M + J_L)\ddot{\theta}_C + 1/N \ (N^2 f_M + f_L)\dot{\theta}_C = T_C \qquad (19)$$

$$T_C = K_1 I_a \qquad (20)$$

$$K_3 E = \dot{I}_a + P I_a \qquad (21)$$

$$E = K_2\theta_R - K_2\theta_C \qquad (22)$$

Substitute equation (22) into equation (21) it becomes:

$$K_2K_3\theta_R - K_2K_3\theta_C = \dot{I}_a + P I_a \qquad (23)$$

Differentiate equation (19) we obtain:

$$\dot{T}_C = 1/N(N^2 J_M + J_L)\dddot{\theta}_C + 1/N \ (N^2 f_M + f_L)\ddot{\theta}_C \qquad (24)$$

and $$\dot{I}_a = 1/K_1(\dot{T}_C) \qquad (25)$$

$$1/K_1N \ (N^2 J_M + J_L)\dddot{\theta}_C + 1/K_1N(N^2 f_M + f_L)\ddot{\theta}_C + P/K_1N(N^2 J_M + J_L)\ddot{\theta}_C +$$

$$P/K_1N(N^2 f_M + f_L)\dot{\theta}_C = K_2K_3\theta_R - K_2K_3\theta_C \qquad (26)$$

Rearranging equation (26) and multiplied both sides by K_1 it becomes:

$$K_1K_2K_3\theta_R = 1/N(N^2J_M + J_L)\dddot{\theta}_C + 1/N(N^2f_M + f_L + N^2PJ_M + PJ_L)\ddot{\theta}_C$$

$$+ P/N(N^2f_M + f_L)\dot{\theta}_C + K_1K_2K_3\theta_C \qquad (27)$$

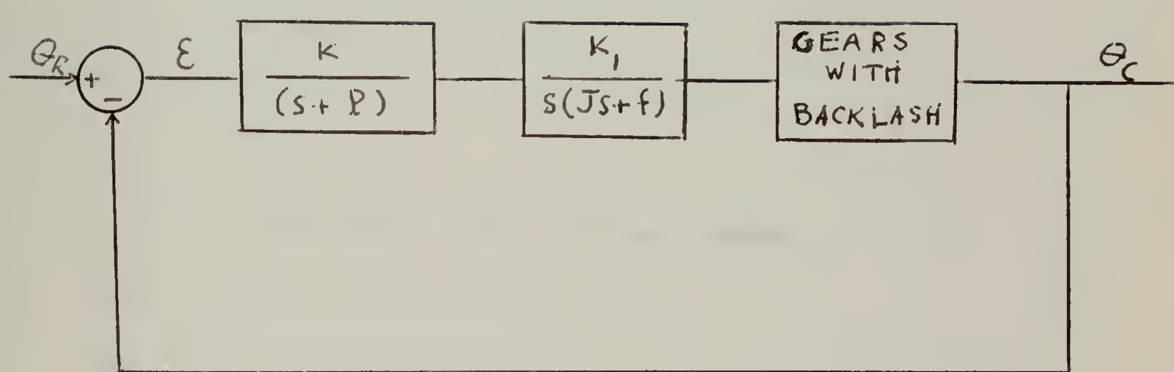

Fig. 12. A block diagram of a third order servo with backlash and load inertia.

Where:

K_a = amplifier constant Volts/Volts

K_1 = motor torque constant ft-lbs/amp.

K_2 = error measurement constant Volts/rad.

K_3 = K_a/L = Volts/Volts-henry

P = R/L = $\frac{\text{Armature resistance}}{\text{Armature inductance}}$ = Ohms/henry

R = Armature resistance = Ohms

L = Armature inductance = henry

N = Gear ration = rad. motor/rad.output

J_M = Inertia of system without load ft-lbs-sec^2/rad.

J_L = Inertia of load ft-lbs-sec^2/rad.

f_M = Friction of system without load

f_L = Friction of load only

J = $N^2 J_M + J_L$ $\quad K = K_2 K_3$

f = $N^2 f_M + f_L$

θ_C = Displacement of the combined system.

θ_L = Displacement of the load only.

θ_R = Input displacement.

Letting N = 1, equation (27) becomes:

(28)

This is a third order differential equation, hence we can not solve it by using phase plane technique. In order to be able to draw the phase plane trajectory we have to solve the above differential equation either by longhand or by using analog computer. Solving equation (28) as a function of time (t),

$$\theta_c = \frac{1/JK_1K}{s^3 + (f/J + P)s^2 + (Pf/J)s + (K_1K/J)}\theta_R + \frac{s^3 + (f/J + P)s^2 + (Pf/J)s}{s[s^3 + (f/J+P)s^2 + (Pf/J)s + (K_1K/J)]}\theta_{co} + \frac{[s + (f/J + P)]\dot{\theta}_{co} + \ddot{\theta}_{co}}{s^3 + (f/J + P)s^2 + (fP/J)s + (K_1K/J)} \tag{29}$$

letting $K_1K = 1$ and $\theta_R = 1/s$, equation (29) becomes:

$$\theta_c = \frac{1/JK_1K + [s^3 + (f/J+P)s^2 + (Pf/J)s]\theta_{co}}{s[s^3 + (f/J+P)s^2 + (Pf/J)s + (K_1K/J)} + \frac{[s + (f/J + P)]\dot{\theta}_{co} + \ddot{\theta}_{co}}{s^3 + (f/J+P)s^2 + (Pf/J)s + 1/J} \tag{30}$$

or
$$\theta_c = \frac{1/J + [s^3 + (f/J+P)s^2 + (Pf/J)s]\theta_{co}}{s[(s+a)(s+b+jc)(s+b-jc)]} + \frac{[s + (f/J+P)]\dot{\theta}_{co} + \ddot{\theta}_{co}}{(s+a)(s+b+jc)(s+b-jc)} \tag{31}$$

where:
$$f/J + P = a + 2b$$
$$fP/J = 2ab + b^2 + c^2$$
$$1/J = a(b^2 + c^2)$$

If initial conditions = 0, equation (31) becomes

$$\theta_c = \frac{1}{J}\left[\frac{1}{a(b^2 + c^2)} + \frac{e^{-at}}{a^2 - 2ab + b^2 + c^2} - \frac{e^{-(b-jc)t}}{2jc(a-b-jc)} + \right.$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$+ \frac{e^{-(b - jc)t}}{2jc(a - b + jc)} \Bigg] u(t) \tag{32}$$

$$\theta_C = \frac{1/J}{(s + a)(s + b + jc)(s + b - jc)} \tag{33}$$

$$\dot{\theta}_C = \frac{1}{J}\left[\frac{e^{-at}}{a^2 - 2ab + b^2 \quad c^2} - \frac{e^{-(b - jc)t}}{2jc(a - b - jc)} + \frac{e^{-(b - jc)t}}{2jc(a - b + jc)}\right] u(t) \tag{34}$$

From equations (32) and (34) , we can draw the phase plane trajectory of the velocity vs. displacement (see Fig: 13)

$$\ddot{\theta}_C = \frac{(1/J)s}{(s + a)(s + b + jc)(s + b - jc)} \tag{35}$$

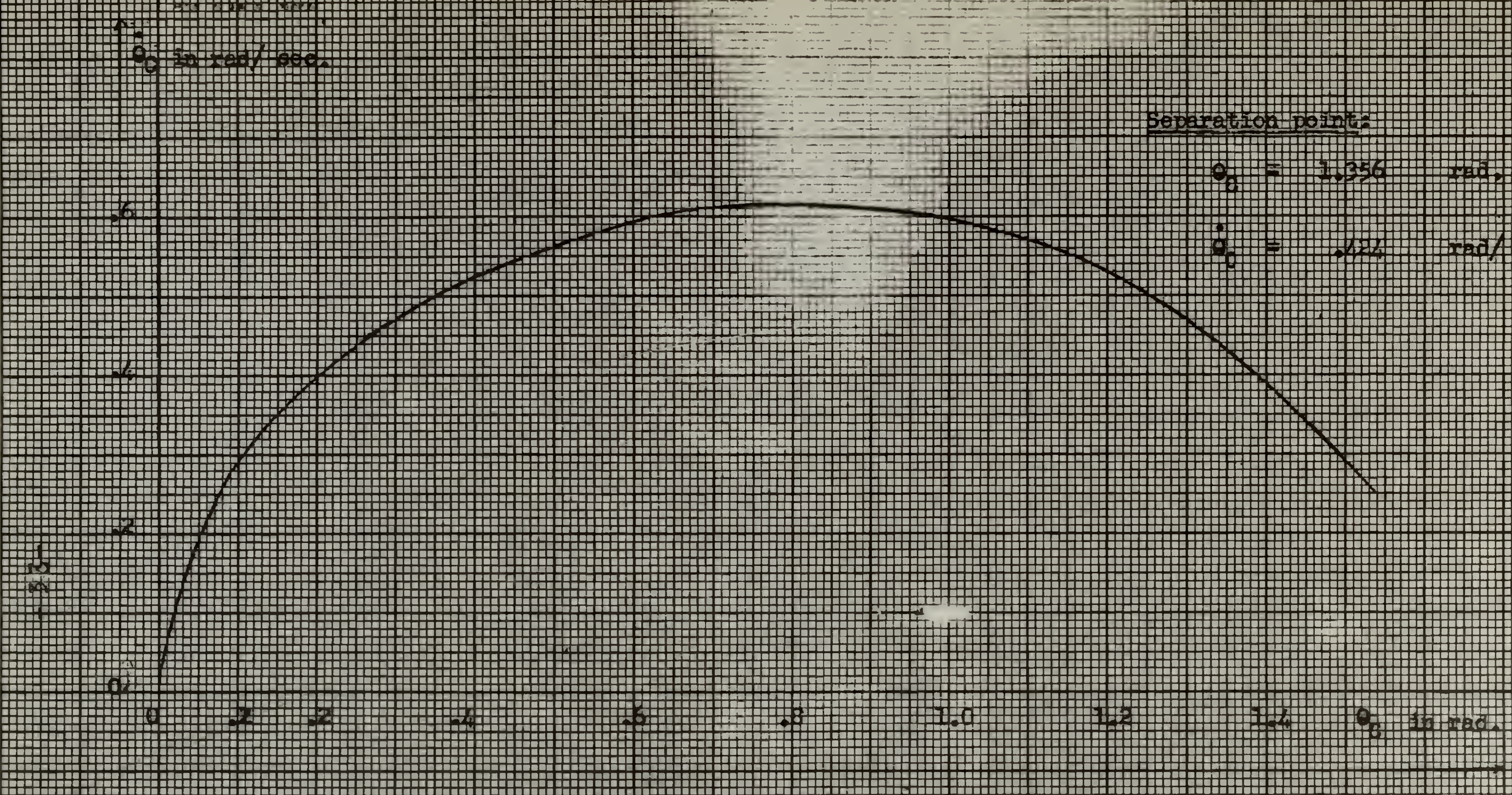

Fig. 13. A phase-plane trajectory of velocity vs displacement for the combined system.

Gain constant = 1.0

Step input = 1.0

Backlash = 1.0

$$\theta_C = \frac{1}{J}\left[\frac{-ae^{at}}{a^2 - 2ab - b^2 - c^2)} + \frac{(b - jc)e^{-(b - jc)t}}{2jc(a - b - jc)} - \frac{(b - jc)e^{-(b - jc)t}}{2jc(a - b - jc)}\right]u(t) \tag{36}$$

Fig. 14 is the plot of acceleration vs. displacement of the combined system.

The equation for the drifting load is determined by the friction and inertia of the load.

$$J_L\ddot{\theta}_L + f_L\dot{\theta}_L = 0 \tag{37}$$

$$\frac{\ddot{\theta}_L}{\dot{\theta}_L} = -\frac{f_L}{J_L} \tag{38}$$

This slope determines the point of seperation.

Solving equation (37) as a function of time (t), it becomes:

$$\theta_L = \frac{1}{s}\,\theta_{Lo} + \frac{1}{s(s + d)}\,\dot{\theta}_{Lo} \tag{39}$$

where

$d = f_L/J_L$

$\theta_{Lo} = \theta_{Co} = \theta_{Mo}$ = initial condition of displacement

$\dot{\theta}_{Lo} = \dot{\theta}_{Co} = \dot{\theta}_{Mo}$ = initial condition of velocity

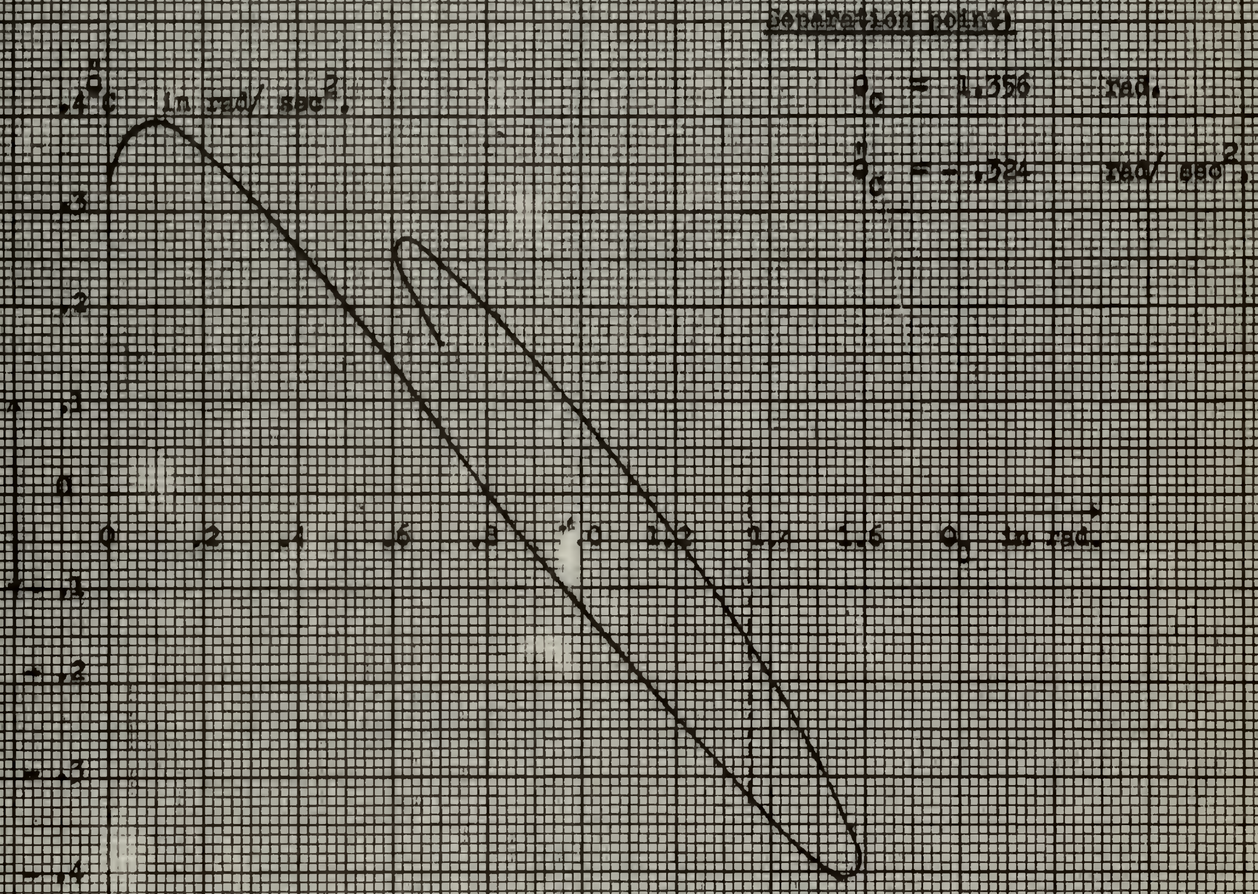

Fig. 14 A plot of acceleration vs. displacement of the combined system.

$$\theta_L = \left[\theta_{Lo} + \frac{1}{d} (1 - e^{-dt}) \dot{\theta}_{Lo} \right] u(t) \tag{40}$$

$$\dot{\theta}_L = \left[\theta_{Lo} + \frac{1}{(s + d)} \theta_{Lo} \right] \tag{41}$$

$$\dot{\theta}_L = \left[(e^{-dt}) \dot{\theta}_{Lo} \right] u(t) \tag{42}$$

After the load has been separated, the equation of the motor becomes:

$$J_M \dddot{\theta}_M + (f_M + J_M P) \ddot{\theta}_M + P f_M \dot{\theta}_M + K_1 K \theta_L = K_1 K \theta_R \tag{43}$$

$$\theta_L = \frac{1}{s} \theta_{Mo} + \frac{1}{s(s + d)} \dot{\theta}_{Mo} \tag{44}$$

Substitute this equation (44) into equation (43) and solve as a function of a s and letting $K_1 K = 1$, it becomes:

$$\theta_M = \frac{1/J_M (1 - \theta_{Mo})}{s^2 (s^2 + (1 + P)s + kP)} + \frac{1/J_M \theta_{Mo}}{s} +$$

$$+ \frac{-1/J_M \theta_{Mo} + (s + k + P)(s + d) \dot{\theta}_{Mo}}{s (s + d)(s^2 + (k + P)s + kP)}$$

$$+ \frac{\ddot{\theta}_{Mo}}{s (s^2 + (k + P) s + kP)} \tag{45}$$

where $k = f_M / J_M$

or

$$\theta_M = \frac{(1 - \theta_{Mo})}{J_M s^2 (s + k)(s + P)} + \frac{\theta_{Mo}}{J_M s} + \frac{\dot{\theta}_M \left[- 1 + J_M (s + k + P)(s + d)\right]}{J_M s(s - d)(s - k)(s - P)} +$$

$$\frac{\ddot{\theta}_{Mo}}{s(s + k)(s + P)} \tag{46}$$

$$\dot{\theta}_M = \frac{(1 - \theta_{Mo})}{J_M s(s + k)(s + P)} + \frac{\theta_{Mo}}{J_M} + \frac{\left[-1 + J_M (s + b + P)(s + d)\right]\dot{\theta}_{Mo}}{J_M(s = d)(s + k)(s + P)}$$

$$+ \frac{\ddot{\theta}_{Mo}}{(s + k)(s + P)} \tag{47}$$

$$\ddot{\theta}_M = \frac{(1 - \theta_{Mo})}{J_M(s + k)(s + P)} + \frac{s\theta_{Mo}}{J_M} + \frac{\left[- 1 + J_M (s + k + P)(s + d)\right] s\dot{\theta}_{Mo}}{J_M(s + d)(s + k)(s + P)}$$

$$+ \frac{s\ddot{\theta}_{Mo}}{(s = k)(s + P)} \tag{48}$$

$$\theta_M = \left[\frac{1}{J_M}\left\{\frac{t}{Ik} - \frac{(k + P)}{(Ik)^2} + \frac{P^2 e^{-kt} - k^2 e^{-Pt}}{(Pk)^2 (P - k)}\right\} +\right.$$

$$+ \frac{\theta_{Mo}}{J_M}\left\{1 - \frac{t}{Pk} - \frac{(k + P)}{(Pk)^2} + \frac{k^2 e^{-Pt} - P^2 e^{-kt}}{(Pk)^2 (P-k)}\right\} +$$

$$+ \frac{\theta_{Mo}}{J_M}\left\{\frac{\left[J_M d(k + P) - 1\right]}{dkP} - \frac{e^{-dt}}{d(k - d)(P - d)}\right\} +$$

$$+ \frac{\theta_{Mo}}{J_M}\left\{- \frac{\left[1 + J_M P(k - d)\right] e^{-kt}}{k(P - k)(k - d)} + \frac{\left[J_M(p - d) - 1\right]}{P(P - d)(p - k)}\right\} +$$

$$+ \theta_{Mo}\left\{\frac{1}{Pk} + \frac{\left[ke^{-Pt} - Pe^{-kt}\right]}{Pk\,(P-k)}\right\}\Bigg]u(t) \qquad (49)$$

$$\dot{\theta}_M = \left[\frac{1}{J_M}\left\{\frac{1}{kP} - \frac{e^{-kt}}{k(P-k)} + \frac{e^{-Pt}}{P(P-k)}\right\} + \right.$$

$$+ \frac{\theta_{Mo}}{J_M}\left\{-\frac{1}{kP} + \frac{e^{-kt}}{k(P-k)} - \frac{e^{-Pt}}{P(P-k)}\right\} +$$

$$+ \frac{\dot{\theta}_{Mo}}{J_M}\left\{\frac{J_M(Pe^{-kt} - ke^{-Pt})}{(P-k)} + \frac{\left[-(P-k)e^{-dt} + (P-d)e^{-kt} + (k-d)e^{-Pt}\right]}{(k-d)\,(P-d)\,(P-k)}\right\}$$

$$\left. + \frac{\ddot{\theta}_{Mo}}{J_M}\left\{\frac{J_M(e^{-kt} - e^{-Pt})}{(P-k)}\right\}\right]u(t) \qquad (50)$$

$$\ddot{\theta}_M = \left[\frac{1}{J_M}\left\{\frac{e^{-kt} - e^{-Pt}}{)P-k)}\right\} - \frac{\theta_{Mo}}{J_M}\left\{\frac{e^{-kt} - e^{-Pt}}{(P-k)}\right\} + \right.$$

$$+ \frac{\dot{\theta}_{Mo}}{J_M}\left\{\frac{J_M(-ke^{-kt} + Pe^{-Pt})}{(P-k)} + \frac{\left[d(P-k)e^{-dt} - k(P-d)e^{-kt} + P(k-d)e^{-Pt}\right]}{(k-d)\,(P-k)\,(P-d)}\right\} +$$

$$\left. + \theta_{Mo}\left\{\frac{-ke^{-kt} + Pe^{-Pt}}{(P-k)}\right\}\right]u(t) \qquad (51)$$

Fig. 15 is the plot of velocity vs. displacement of the motor and the plot of displacement of load vs. displacement of the motor in the backlash region.

From the curve of the load-displacement vs. the motor-displacement we can determine the point of recombination for the motor. Knowing the displacement of the motor, the velocity of the motor at recombination can also be determined.

Applied the law of conservation of momentum:

$$(J_M + J_L)\ \dot{\theta}_C = J_L\dot{\theta}_L + J_M\dot{\theta}_M \qquad (52)$$

From the above equation $\dot{\theta}_L$ and $\dot{\theta}_M$ are known quantities, so we can evaluate the value of $\dot{\theta}_C$.

For the determination of the initial condition of the velocity at the point of recombination we have to apply the torque-equation of the motor.

$$T_M = J_M\ddot{\theta}_M + f_M\dot{\theta}_M \qquad (53)$$

Multiply equation (51) by J_M and equation (50) by f_L and add it together we obtain:

$$T_M = \left[\left\{\frac{e^{-kt} - e^{-Pt}}{(P - k)}\right. - \theta_{Mo}\left\{\frac{(e^{-kt} - e^{-Pt})}{(P - k)}\right\} + \right.$$

$$+ \dot{\theta}_{Mo} J_M \left\{ \frac{(ke^{-kt}+Pe^{-Pt})}{(P-k)} + \frac{d(P-k)e^{-dt}-k(P-d)e^{-kt}+P(k-d)e^{-Pt}}{J_M(k-d)(P-d)(P-k)} \right\} +$$

$$+ \ddot{\theta}_{Mo} J_M \left\{ \frac{-ke^{-kt}+Pe^{-Pt}}{(P-k)} \right\} + \left\{ \frac{1}{P} + \frac{(-Pe^{-kt}+ke^{-Pt})}{(P-k)P} \right\} +$$

$$+ \theta_{Mo} \left\{ -\frac{1}{P} + \frac{(Pe^{-kt}-ke^{-Pt})}{(P-k)P} \right\} +$$

$$+ \dot{\theta}_{Mo} J_M \left\{ \frac{k(Pe^{-kt}-ke^{-Pt})}{(P-k)} + \frac{\left[-k(P-k)e^{-dt}+k(P-d)e^{-kt}+k(k-d)e^{-Pt}\right]}{J_M(k-d)(P-d)(P-k)} + \right\}$$

$$+ \dddot{\theta}_{Mo} J_M \left\{ \frac{k(e^{-kt}-e^{-Pt})}{(P-k)} \right\} \Bigg] u(t) \tag{54}$$

Fig. 16 is the plot of the motor-torque vs. the motor-displacement. Knowing the motor-torque and the motor velocity at the point of recombination, we can determine the motor-acceleration at that point from equation (53).

From here on we wolve for equation (32) using the initial condition: θ_{Co}, $\dot{\theta}_{Co}$ and $\ddot{\theta}_{Co}$, until it hit the point of separation again.

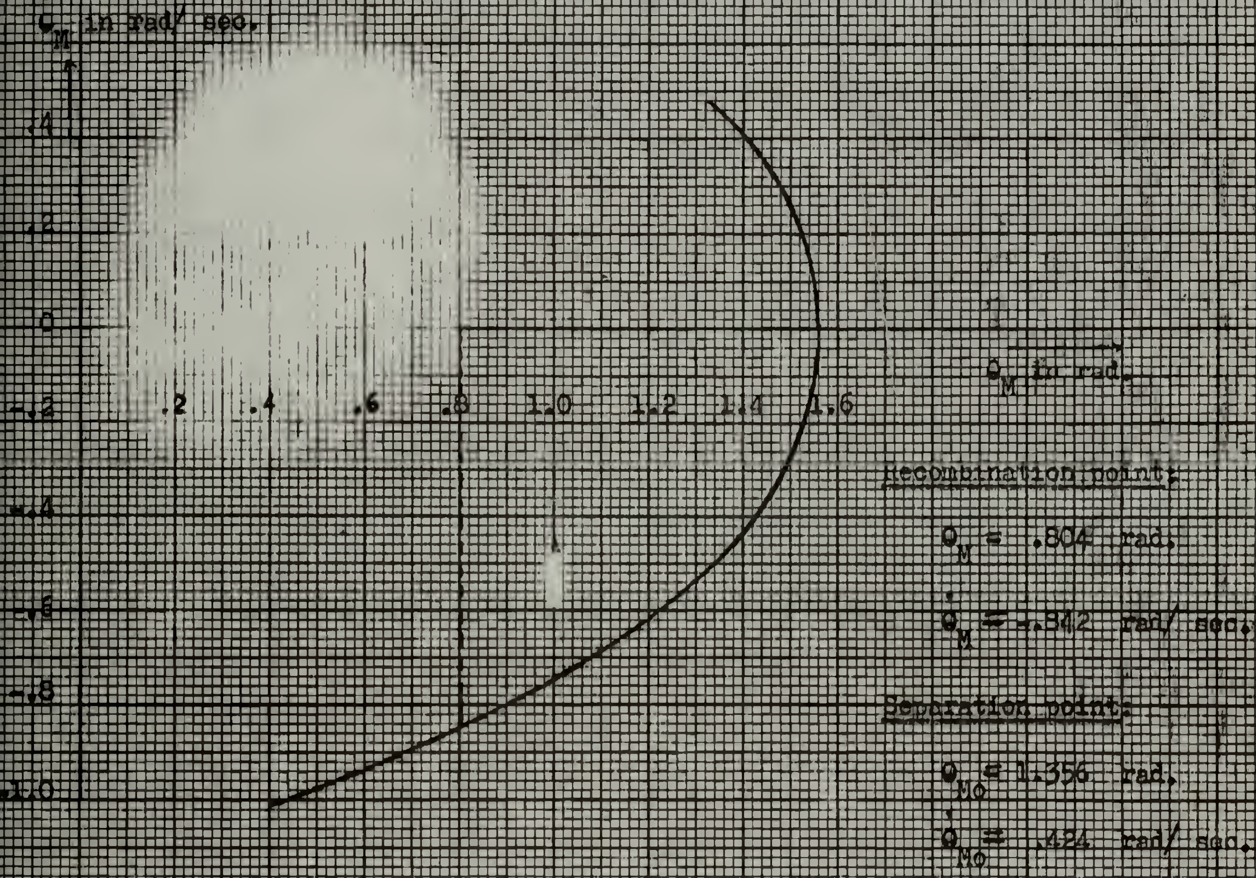

Fig. 15. A plot of motor velocity vs. motor displacement after separation.

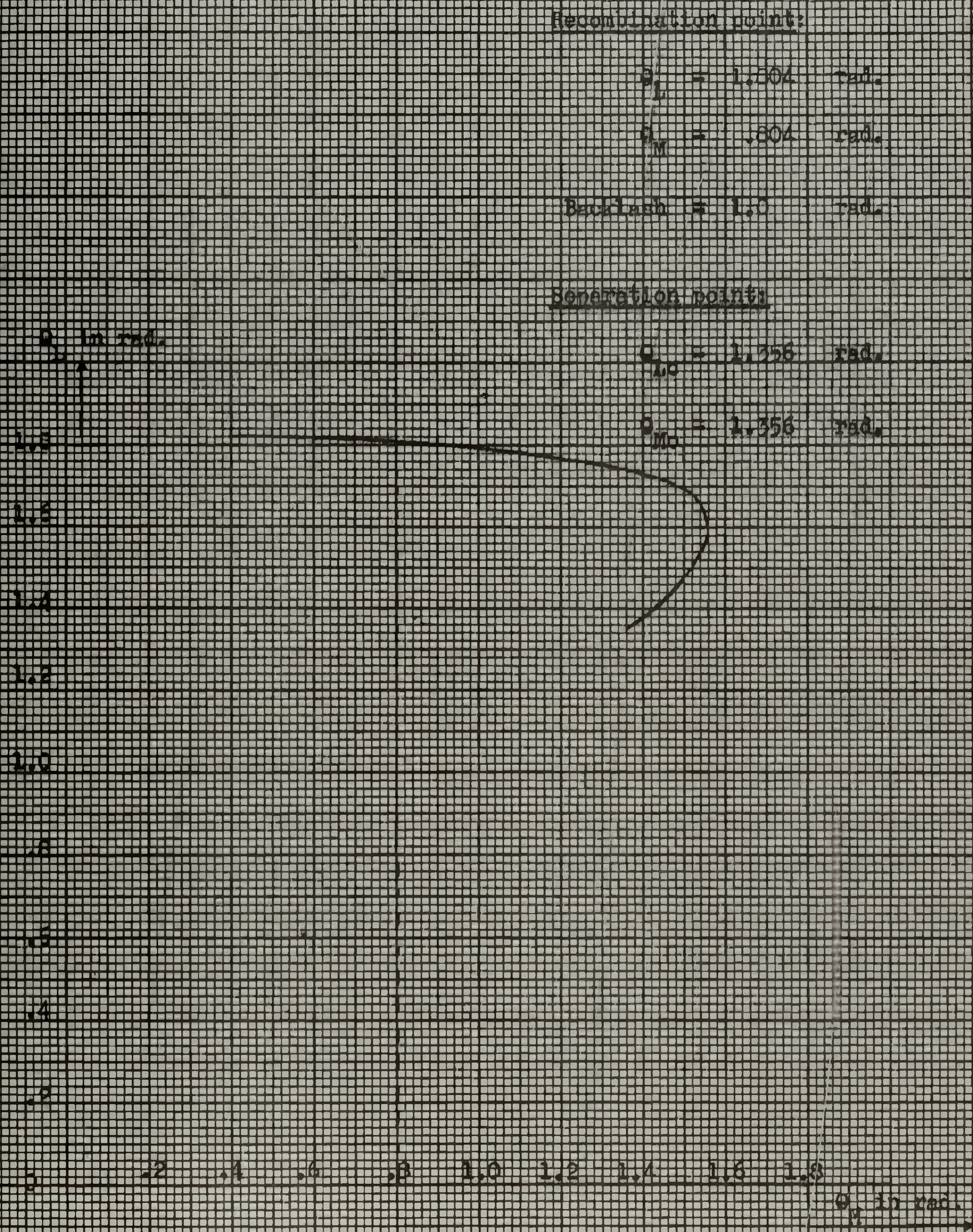

Fig.15. A plot of load displacement vs. motor displacement after separation.

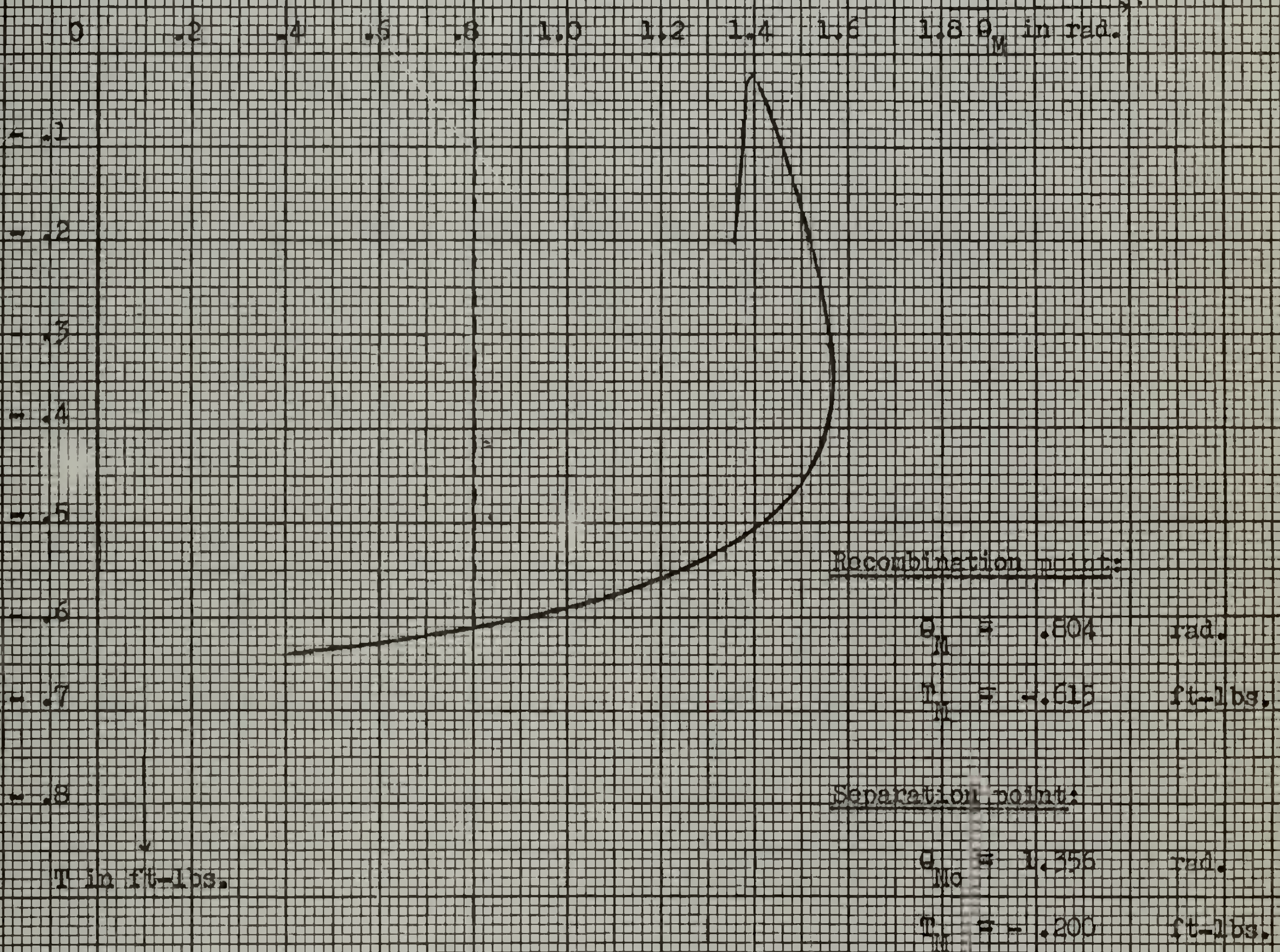

Fig:17. A plot of motor torque vs. motor displacement after separation.

$$G(s) = \frac{K}{s(s + .9 + j\,.436)(s + .9 - j\,.436)}$$

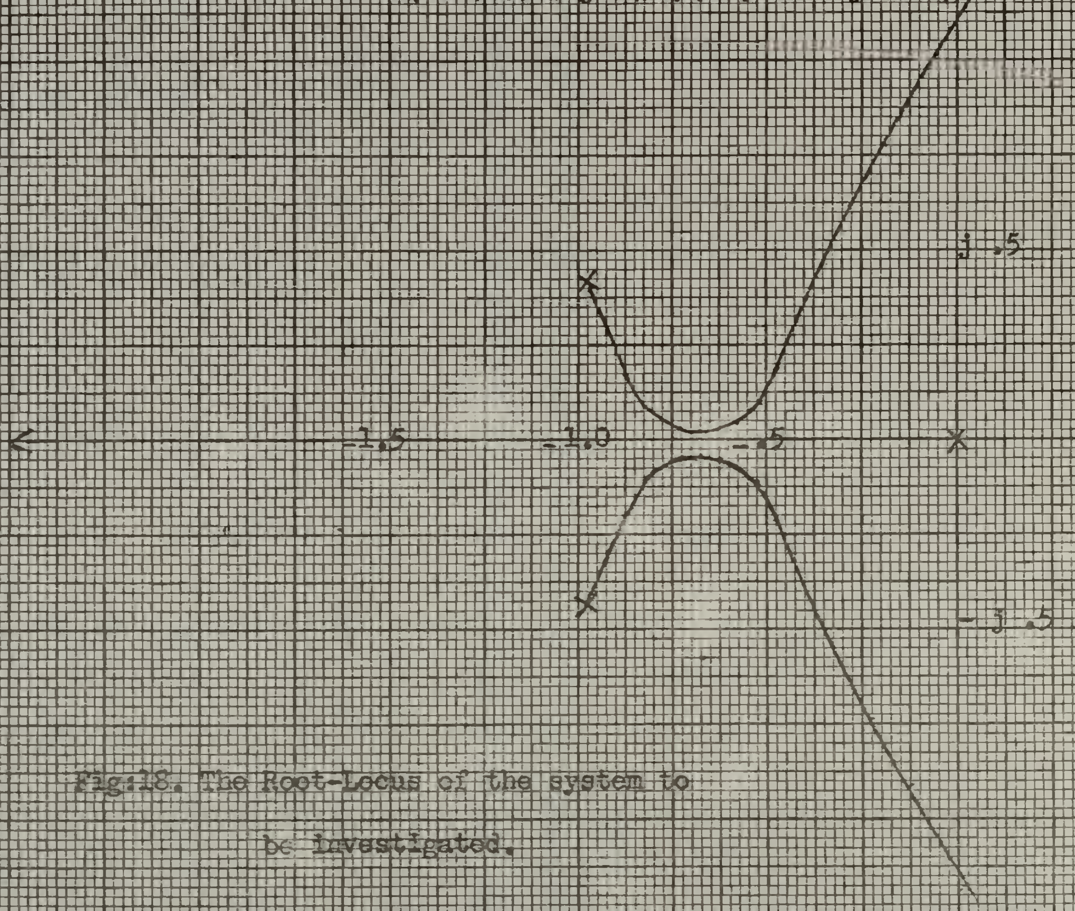

Fig:18. The Root-Locus of the system to be Investigated.

Case I:

For the combined system

$$\dddot{\theta}_C + 1.8\ddot{\theta}_C + \dot{\theta}_C + \theta_C = 1.0 \qquad (55)$$

Where			
	$J = 1.0$	$f = 1.0$	$N = 1.0$
	$J_M = .6$	$f_M = .68$	$P = 1.0$
	$J_L = .4$	$f_L = .32$	$K_1K = 1.0$
			$\theta_R = 1.0$

$$G(s) = \frac{1}{s\,(s^2 + 1.8s + 1.0)} = \frac{1}{s(s + .9 + j.436)\,(s + .9 - .j436)}$$

Fig. 17 shows the root locus of the system to be investigated. To solve equation (55), used an analog computer and the result is plotted in Fig. (18).

The load separated with the equation:

$$.4\ddot{\theta}_L + .32\dot{\theta}_L = 0 \qquad (56)$$

The slope of the load is

$$N = \frac{\ddot{\theta}_L}{\dot{\theta}_L} = -.80 \qquad (57)$$

Equations (56) is plotted in Fig. (18).

The equation of the motor after separation:

$$\dddot{\theta}_M + 2.467\ddot{\theta}_M + 1.133\dot{\theta}_M + 1.667\theta_L = 1.667 \qquad (58)$$

From the slope of the load we can find the initial condition for equation (58). In our case these are:

$$\theta_{Co} = \theta_{Lo} = \theta_{Mo} = 1.360$$

$$\dot{\theta}_{Co} = \dot{\theta}_{Lo} = \dot{\theta}_{Mo} = .424$$

$$\ddot{\theta}_{Co} = \ddot{\theta}_{Lo} = \ddot{\theta}_{Mo} = .324$$

Using these initial conditions we solve for equation (58) and the result is plotted in Fig. (18).

From Fig. (23) for a backlash = .42 rad.

$\theta_L = .328$ rad $\qquad \dot{\theta}_L = -.074$ rad/sec.

$\theta_M = .748$ rad $\qquad \dot{\theta}_M = .480$ rad/sec.

$T_M = .450$ ft-lbs.

$\dot{\theta}_C = .6(.480) - .4(.074) = .258$ rad/sec.

$\ddot{\theta}_C = .450 - .258 = .192$ rad/sec^2.

From Fig.(28), for $\dot{\theta}_C = .258$ rad/sec, $\theta_C = .328$ rad.

Initial conditions are:

$$\theta_{Co} = .328 \text{ rad.}$$

$$\dot{\theta}_{Co} = .258 \text{ rad/sec.}$$

$$\ddot{\theta}_{Co} = .192 \text{ rad/sec}^2.$$

Using these as initial conditions we solve for equation (55) again. Values of θ_C, $\dot{\theta}_C$ and $\ddot{\theta}_C$ at points of separation are tabulated on table (5), with several values of backlash.

TABLE 5

θ_C	$\dot{\theta}_C$	$\ddot{\theta}_C$	Backlash
1.360	.424	-.324	1.0
.600	-.430	.336	1.0
1.377	.444	-.342	1.0
1.360	.424	-.324	.9
.620	-.420	.326	.9
1.359	.420	-.320	.9
1.360	.424	-.324	.8
.639	-.428	.308	.8
1.348	.408	-.320	.8
.666	-.396	.294	.8
1.321	.380	-.300	.8
.670	-.390	.270	.8
1.320	.376	-.298	.8
1.360	.424	-.324	.7
.652	-.412	.310	.7
1.332	.392	-.304	.7

TABLE 5 continued

θ_C	$\dot{\theta}_C$	$\ddot{\theta}_C$	Backlash
1.360	.424	-.324	.6
.660	-.396	.292	.6
1.306	.356	-.286	.6
.728	-.316	.224	.6
1.256	.300	-.234	.6
1.360	.424	-.324	.5
.666	-.394	.288	.5
1.286	.334	-.270	.5
.736	-.312	.226	.5
1.240	.280	-.222	.5
.768	-.276	.192	.5
1.220	.254	-.204	.5
.788	-.250	.186	.5
1.200	.230	-.200	.5
.800	-.240	.170	.5
1.192	.220	-.188	.5
.810	-.230	.165	.5

TABLE 5 continued

θ_C	$\dot{\theta}_C$	$\ddot{\theta}_C$	Backlash
1.360	.424	-.324	.4
.676	-.382	.264	.4
1.260	.306	-.244	.4
.754	-.290	.200	.4
1.208	.246	-.194	.4
.798	-.240	.168	.4
1.164	.216	-.166	.4
.825	-.208	.144	.4
1.140	.200	-.142	.4
.832	-.196	.118	.4
1.120	.194	-.134	.4
1.360	.424	-.324	.3
.705	-.344	.264	.3
1.240	.280	-.234	.3
.800	-.234	.172	.3
1.175	.208	-.160	.3
.846	-.180	.134	.3
1.155	.184	-.140	.3
.852	-.170	.120	.3
1.132	.158	-.120	.3
.864	-.158	.110	.3

TABLE 5 continued

θ_C	$\dot{\theta}_C$	$\ddot{\theta}_C$	Backlash
1.360	.424	-.324	.2
.720	-.344	.254	.2
1.222	.276	-.210	.2
.820	-.220	.198	.2
1.136	.170	-.120	.2
.880	-.148	.108	.2
1.098	.124	-.100	.2
1.360	.424	-.324	.1
.740	-.312	.238	.1
.1200	.236	-.180	.1
.860	-.190	.140	.1
1.120	.134	-.112	.1
.910	-.110	.094	.1
1.074	.092	-.064	.1
.940	-.080	.052	.1

Values of θ_L, $\dot{\theta}_L$, θ_M, $\ddot{\theta}_M$, $\dot{\theta}_C$, $\ddot{\theta}_C$ and T_M at points of recombination are tabulated on Table 6 for several values of backlash.

TABLE 6

θ_L	$\dot{\theta}_L$	θ_M	$\dot{\theta}_M$	T_M	$\dot{\theta}_C$	$\ddot{\theta}_C$	Backlash
1.804	.070	.804	-.842	-.615	-.477	-.233	1.0
.124	-.050	1.124	.904	.632	.522	.214	1.0
1.840	.076	.840	-.852	-.636	-.481	-.251	1.0
1.798	.074	.898	-.800	-.604	-.4504	-.244	.9
.170	-.060	1.700	.790	.606	.450	.246	.9
1.794	.670	.894	-.788	-.590	-.445	-.234	.9
1.790	.080	.990	-.750	-.596	-.418	-.262	.8
.86	-.072	.986	.746	.584	.423	.246	.8
1.784	.082	1.084	-.692	-.582	-.383	-.275	.7
.206	-.056	.906	.664	.560	.376	259	.7
1.734	.072	1.034	-.680	-.548	-.379	-.245	.7
1.776	.090	1.176	-.632	-.550	-.343	-.280	.6
.258	-.072	.858	.610	.526	.3372	.2562	.6
1.670	.068	1.070	-.376	-.350	-.1984	-.1912	.6
.398	-.054	.998	.554	.440	.3108	.1914	.6
1.558	.058	.958	-.539	-.422	-.2996	-.1824	.6
1.762	.104	1.262	-.560	-.552	-.294	-.316	.5
.274	-.080	.774	.544	.494	.2944	.2584	.5
1.602	.082	1.102	-.512	-.448	-.2744	-.2286	.5
.404	-.046	.904	.488	.406	.2744	.1864	.5
1.508	.066	1.008	-.484	-.390	-.264	-.1788	.5
.472	-.042	.972	.460	.370	.2592	.1626	.5
1.472	.055	972	-.476	-.364	-.2636	-.1332	.5
.512	-.034	1.012	.448	.334	.2552	.1298	.5
1.430	.048	.930	-.434	-.324	-.2412	-.131	.5
.536	-.032	1.036	.422	.328	.2404	.1376	.5
1.414	.041	.914	-.426	-.310	-.2364	-.1208	.5

Figs. 18, 19, 20, 21, 22, 23, 24, 25, 26, and 27 show us the result of the combined trajectories with backlash from .1 till 1.0rad. In our case limit cycle occurs. Values of limit cycles versus backlash is tabulated on Table 7. From the table we come to the conclusion that, if we increase the amount of backlash the amplitude of the limit cycle will increase too.

TABLE 7

For Gain-constant is 1.0

Backlash	Amplitude of limitcycle
.1	---
.2	---
.3	---
.4	.355
.5	.435
.6	.5625
.7	.725
.8	.760
.9	.810
1.0	.840

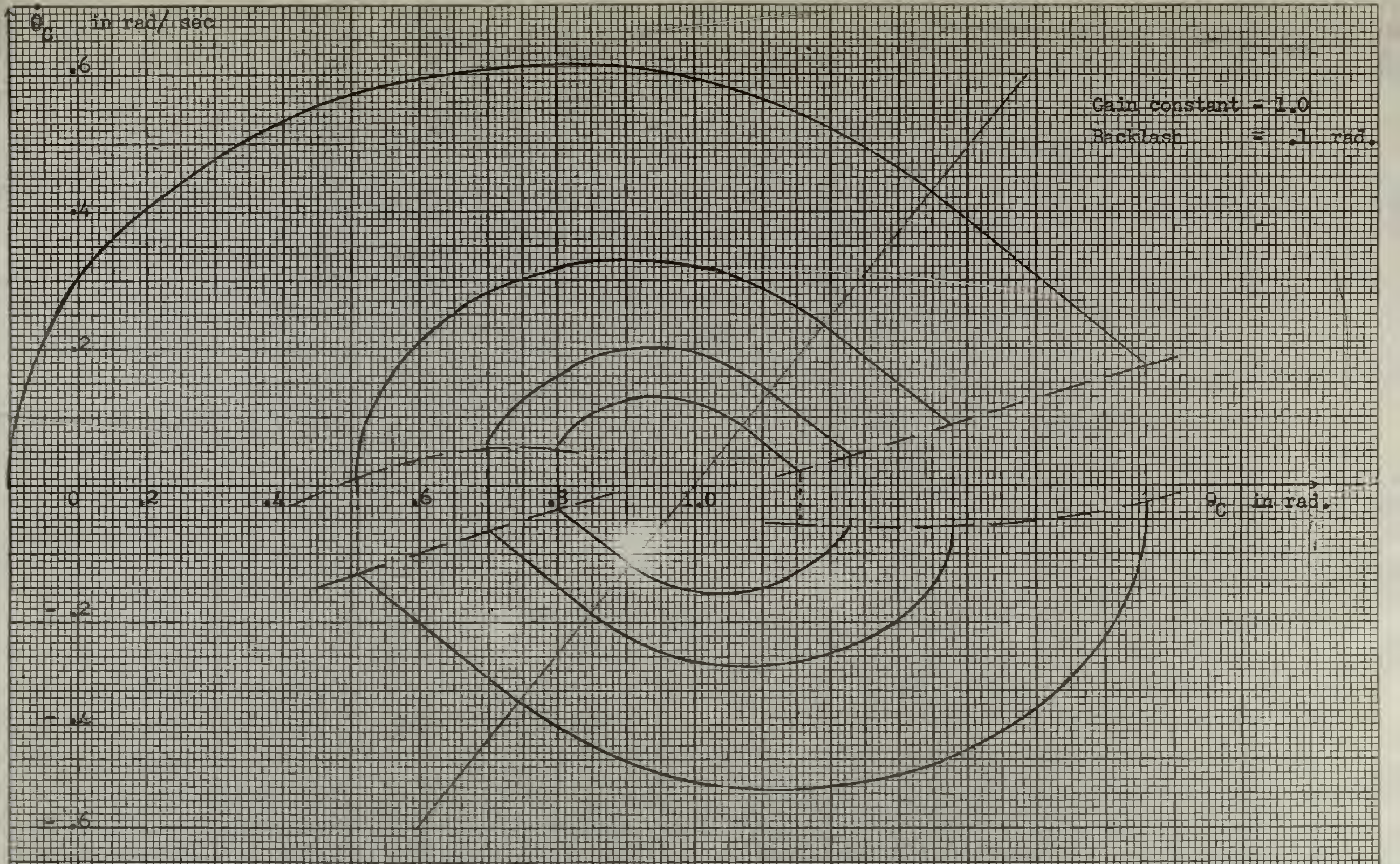

Fig. 19 A complete phase plane trajectory of the combined system.

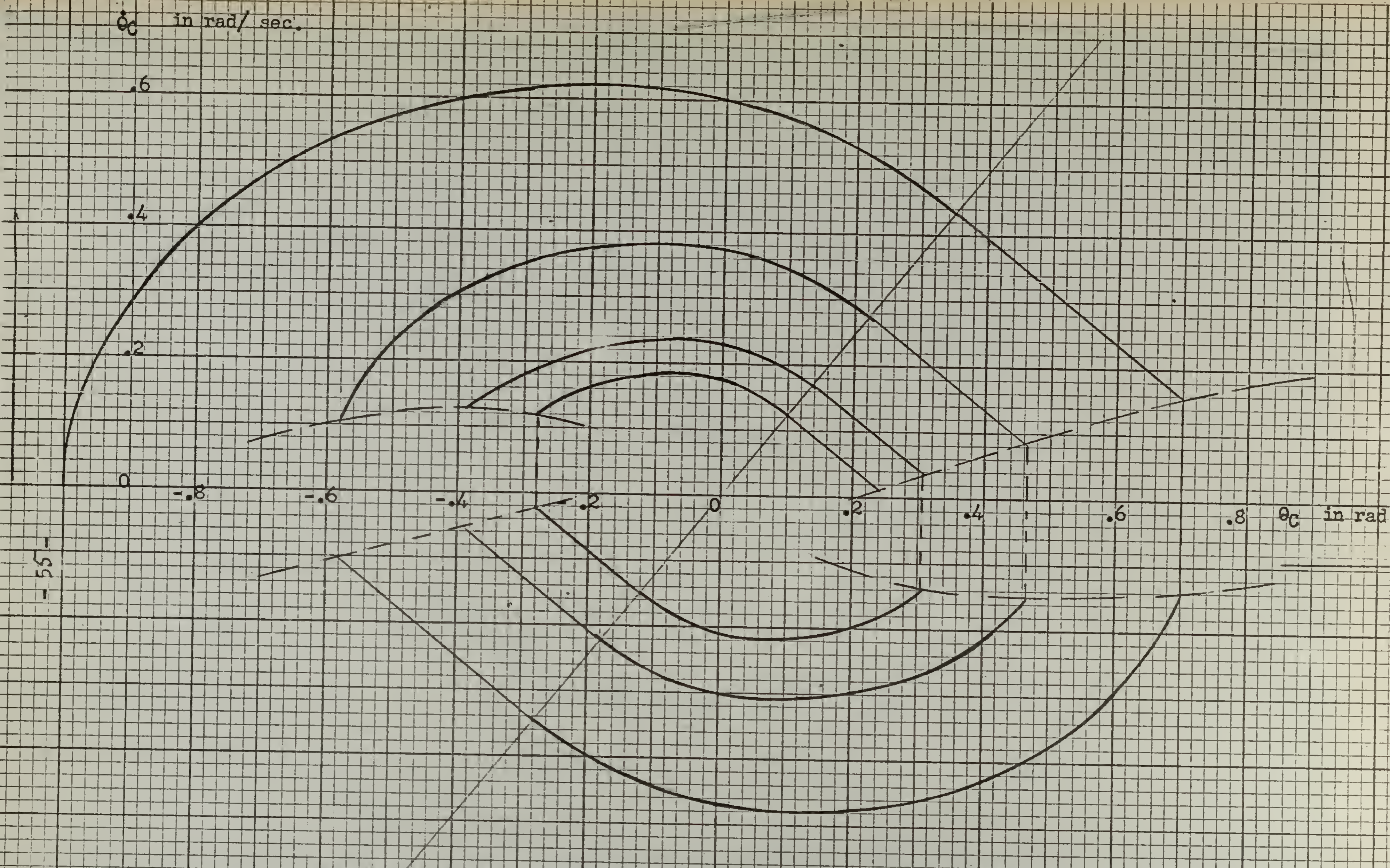

Fig:20. A complete phase plane trajectory of the combined system

with Gain-constant = 1.0 , Unity step-input and Backlash = .2 rad.

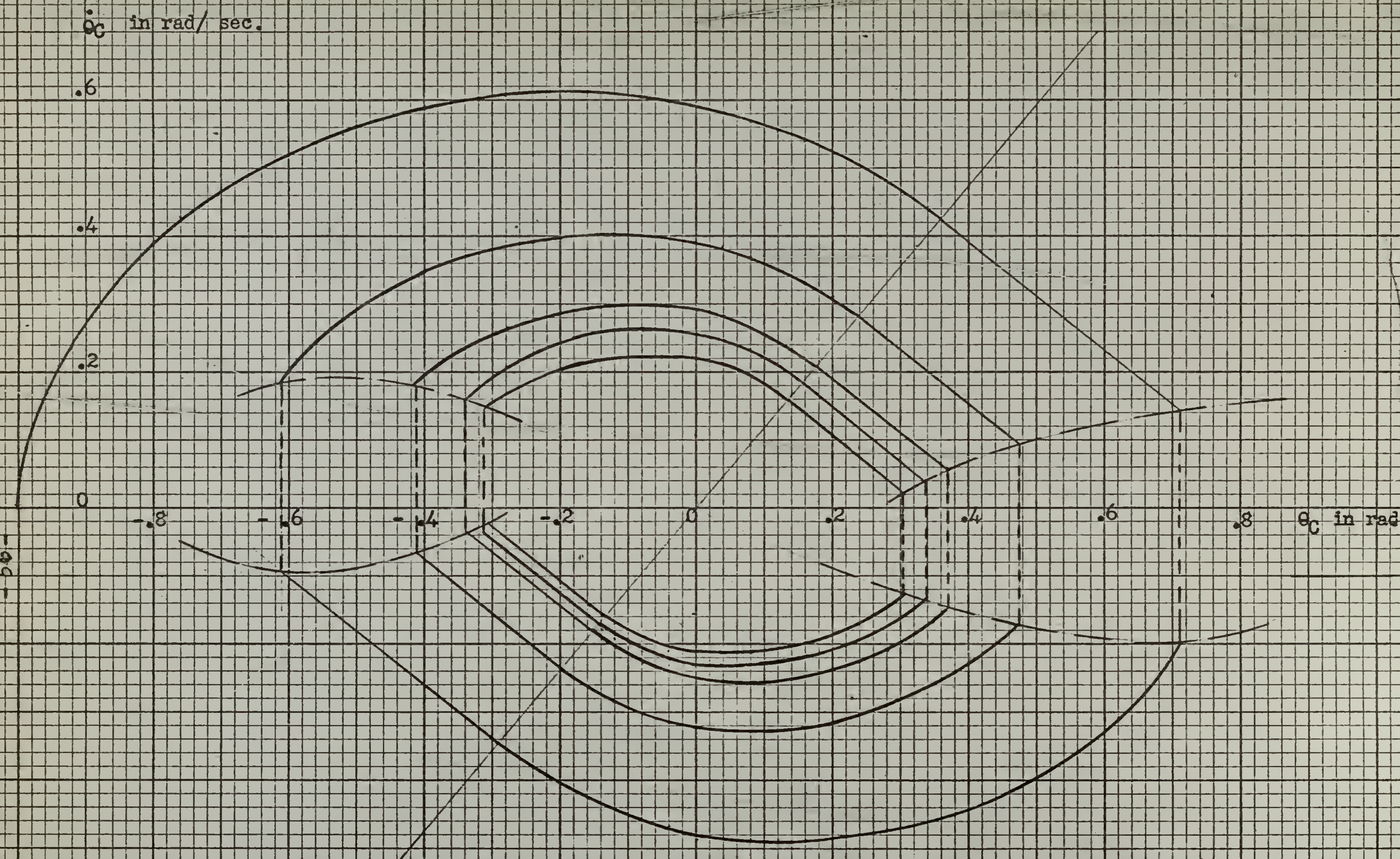

Fig: 21 A complete phase plane trajectory of the combined system with Gain-constant = 1.0, Unity step-input and Backlash = .3 rad.

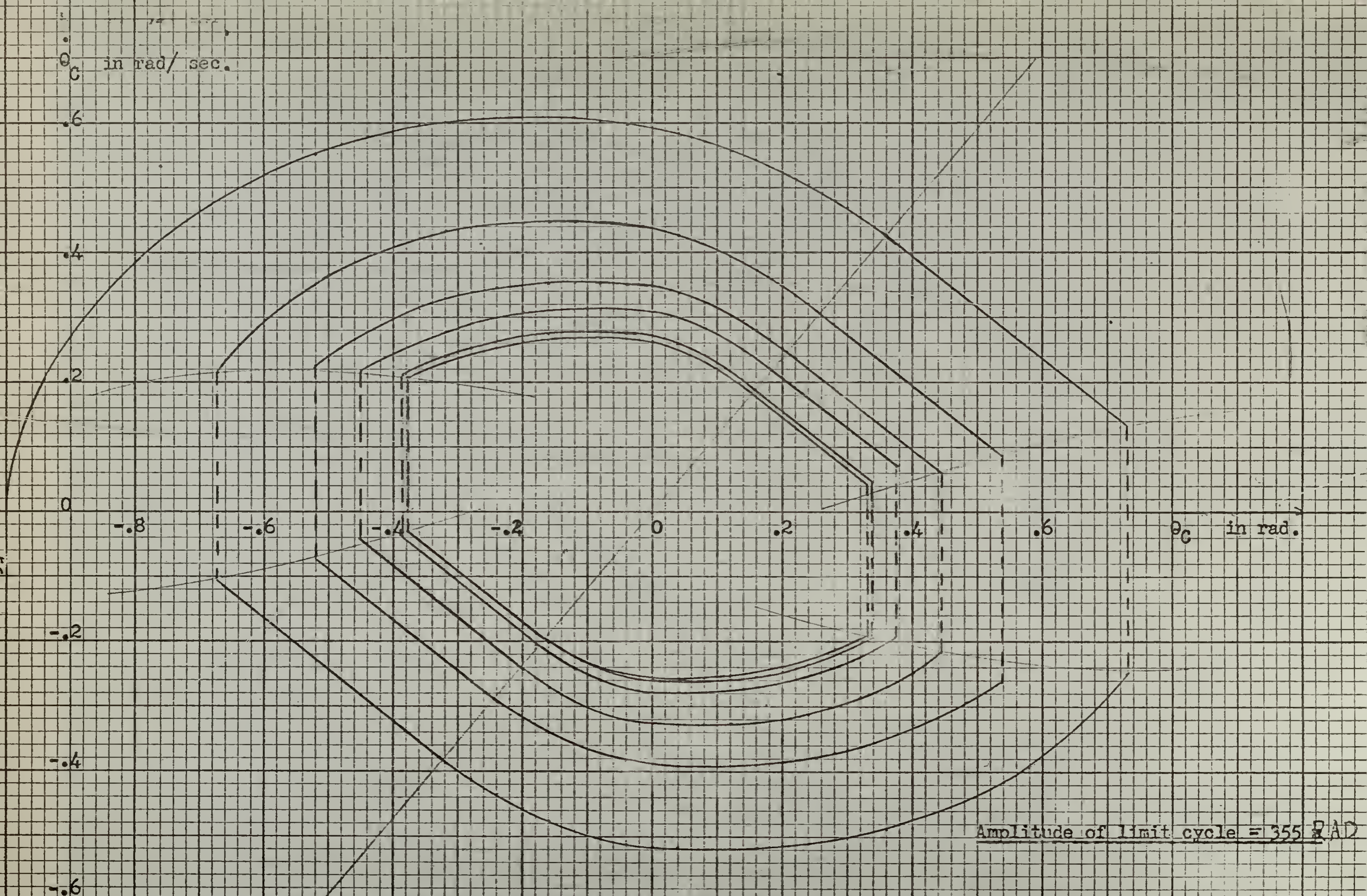

Fig: 22 A complete phase plane trajectory of the combined system with Gain constant = 1.0, Unity step-input and Backlash = .4 rad.

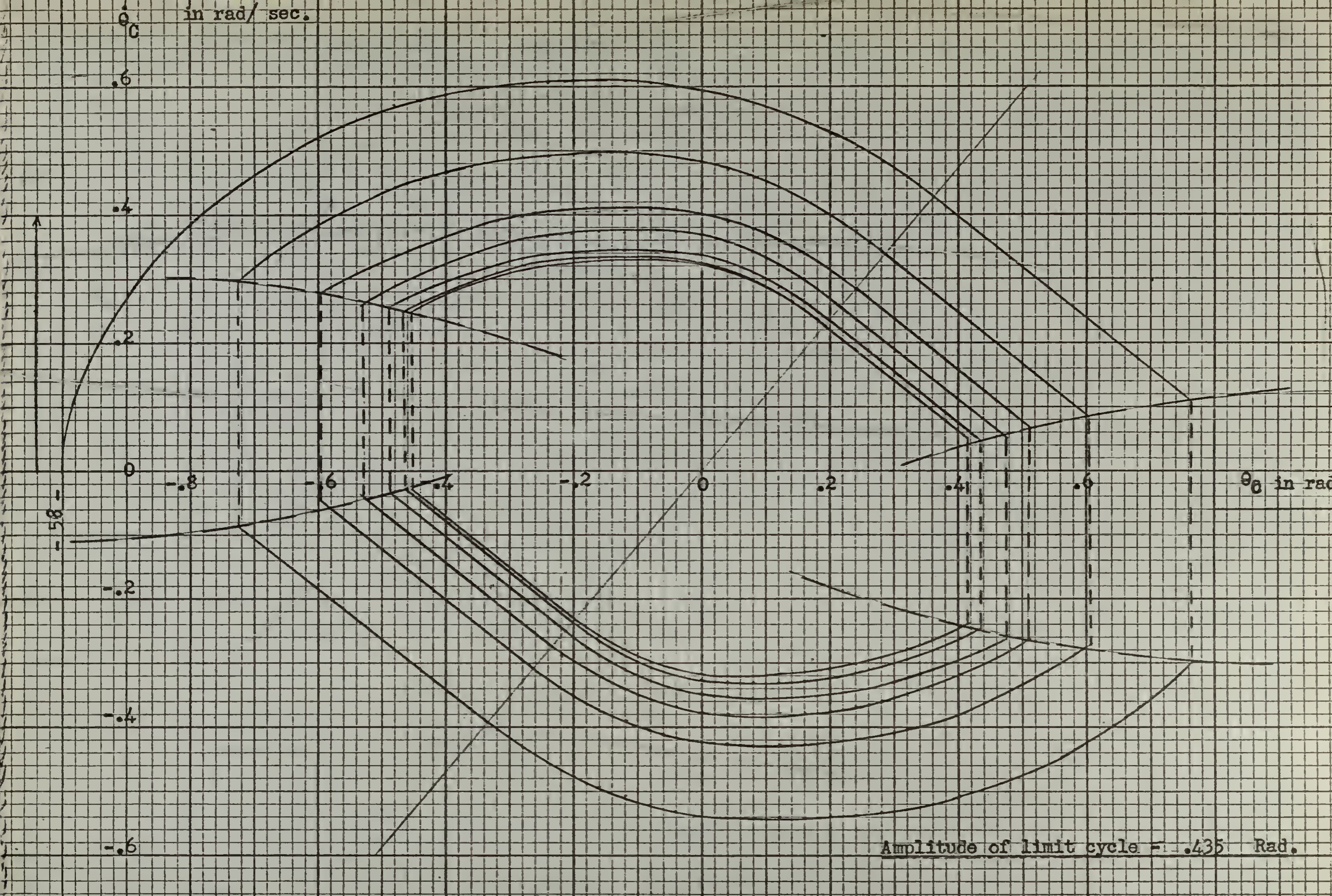

Fig: 23 A complete phase plane trajectory of the combined system with Gain constant = 1.0, Unity step-input and Backlash = .5 rad.

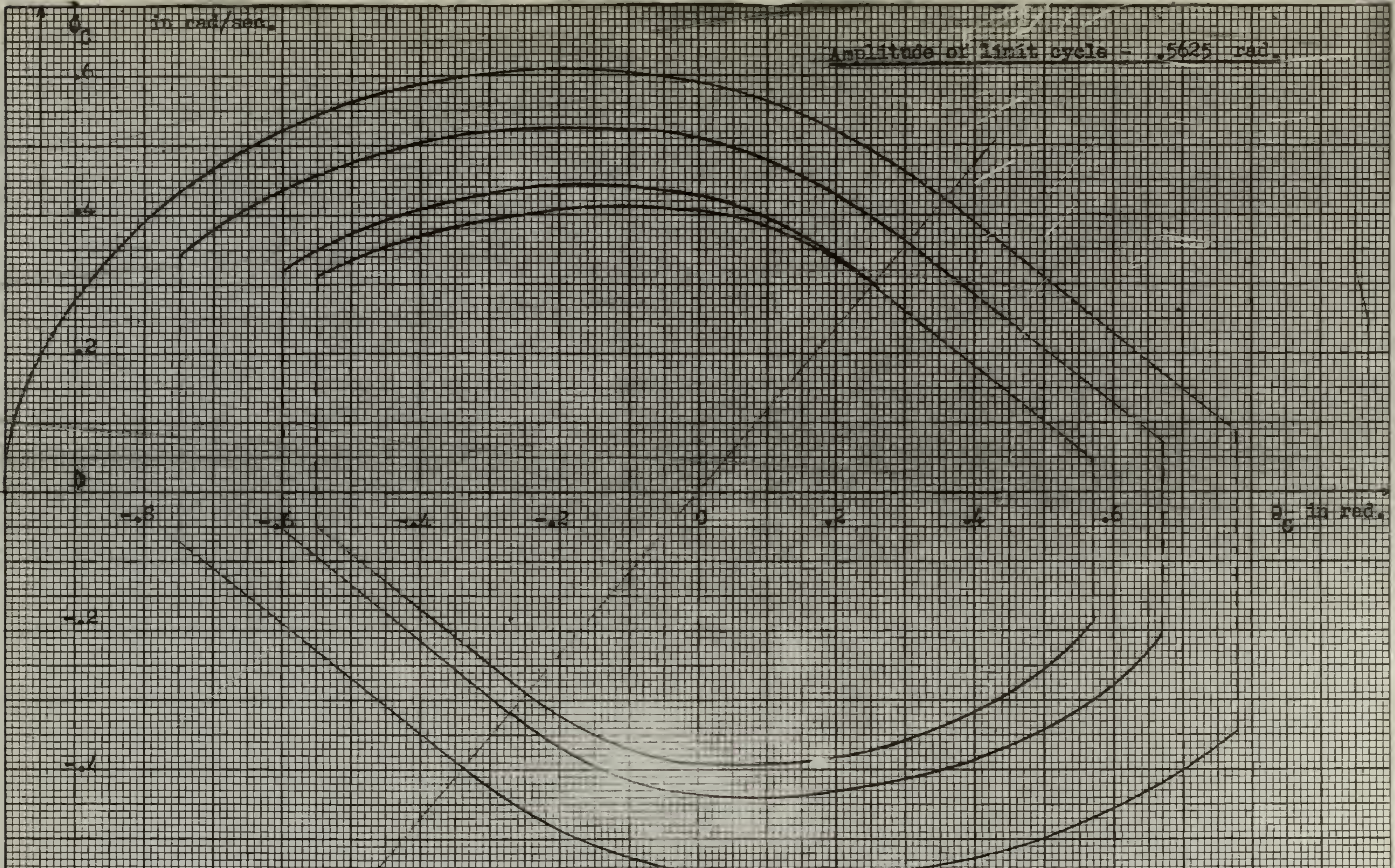

Fig: 24 A complete phase plane trajectory of the combined system

with Gain constant = 1.0, Unity step-input and Backlash = .6 rad.

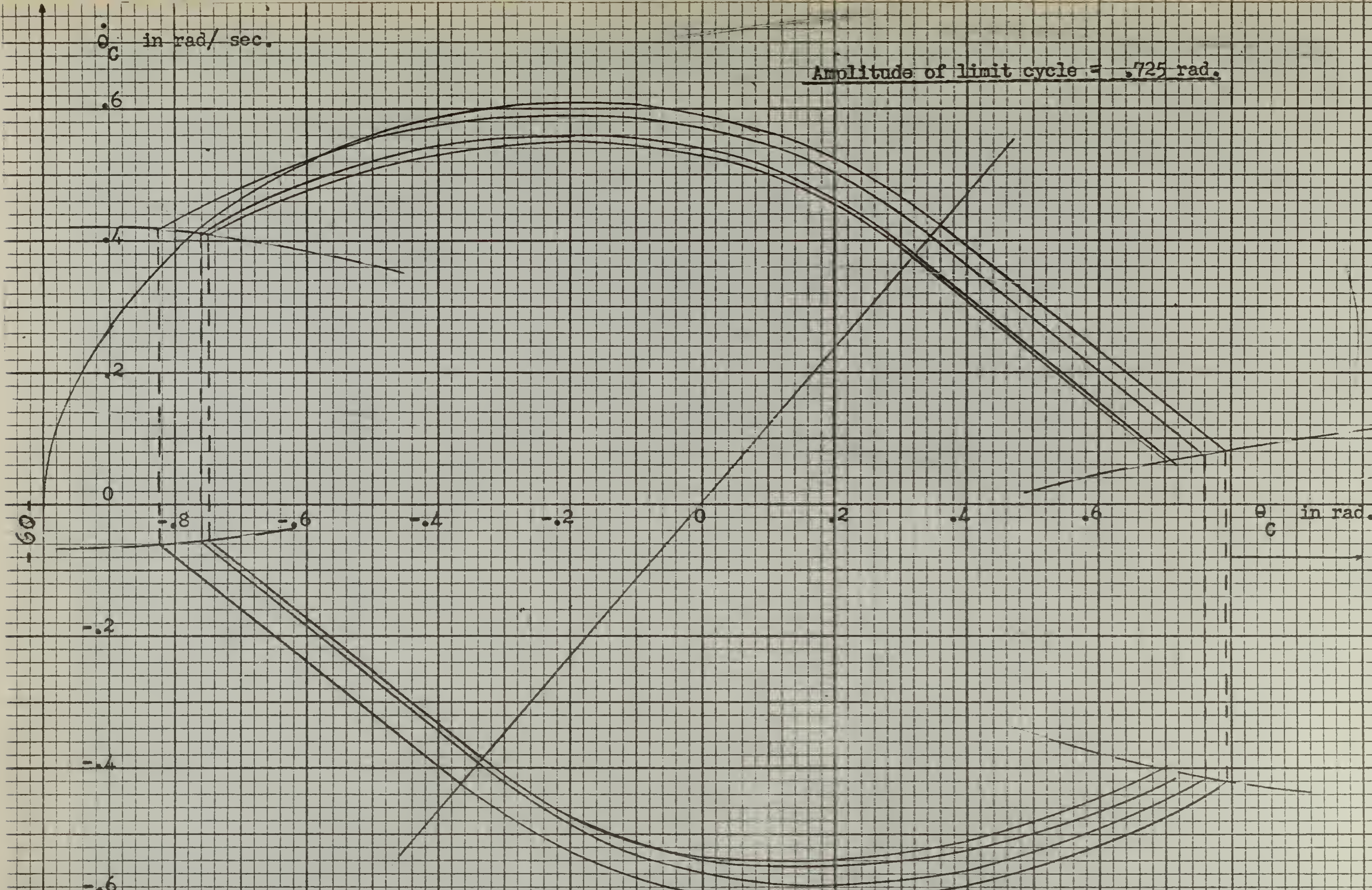

Fig: 25 A complete phase plane trajectory of the combined system

with Gain constant = 1.0, Unity step-input and Backlash = .7 rad.

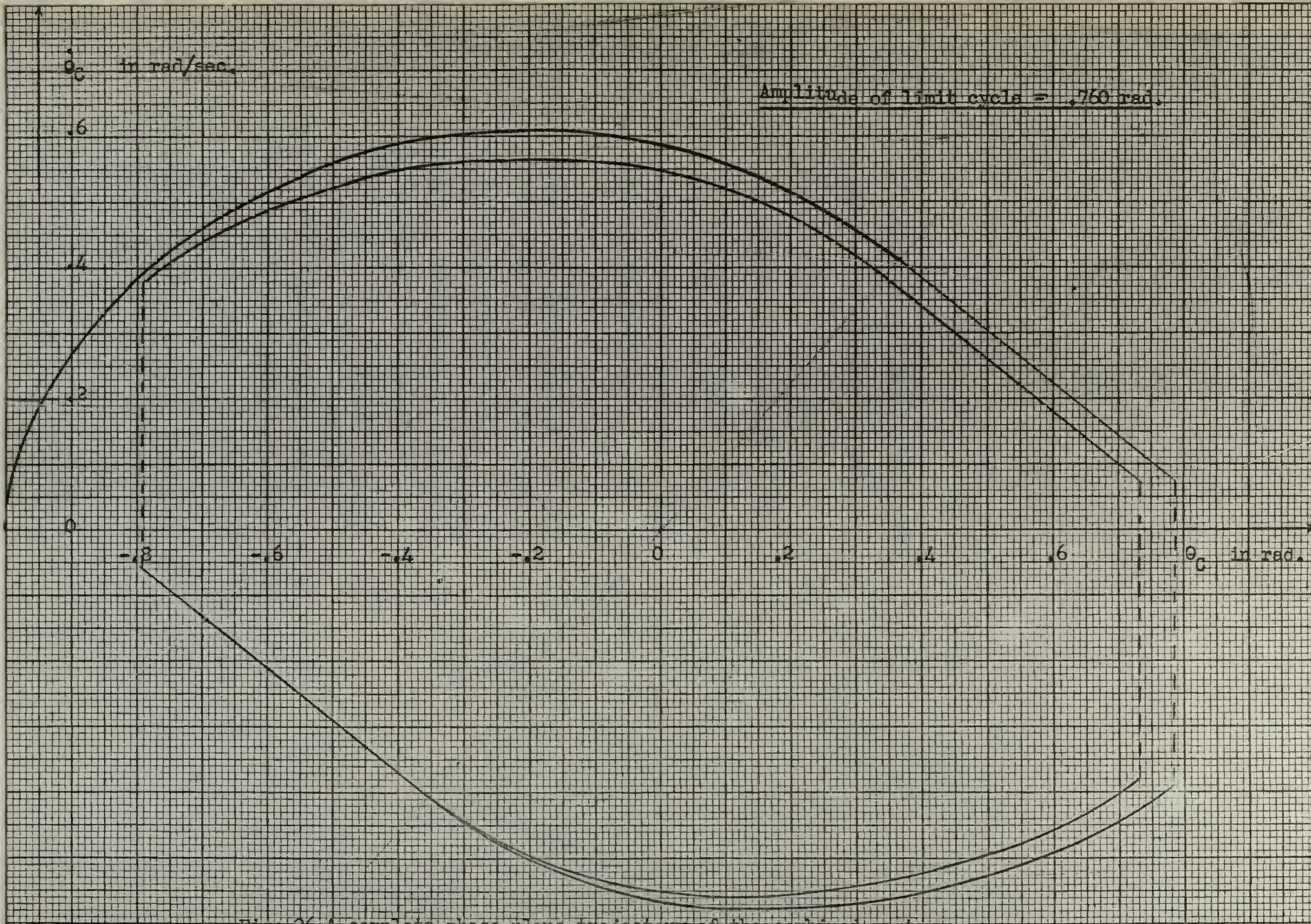

Fig: 26 A complete phase plane trajectory of the combined system
with Gain constant = 1.0, Unity step-input and Backlash = .8 rad.

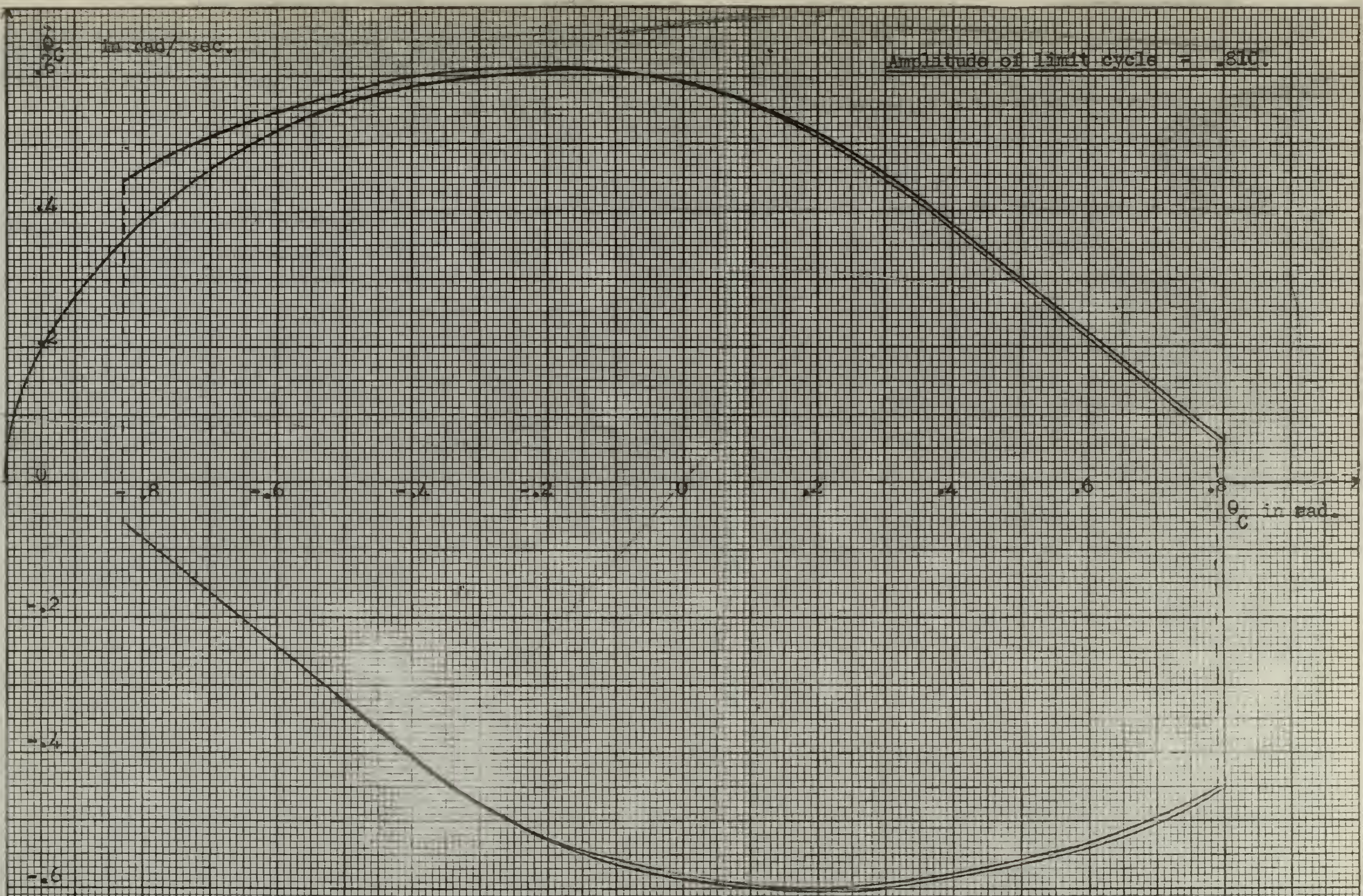

Fig. 27 A complete phase plane trajectory of the combined system

with Gain constant = 1.0, Unity step input and Backlash = .9 rad,

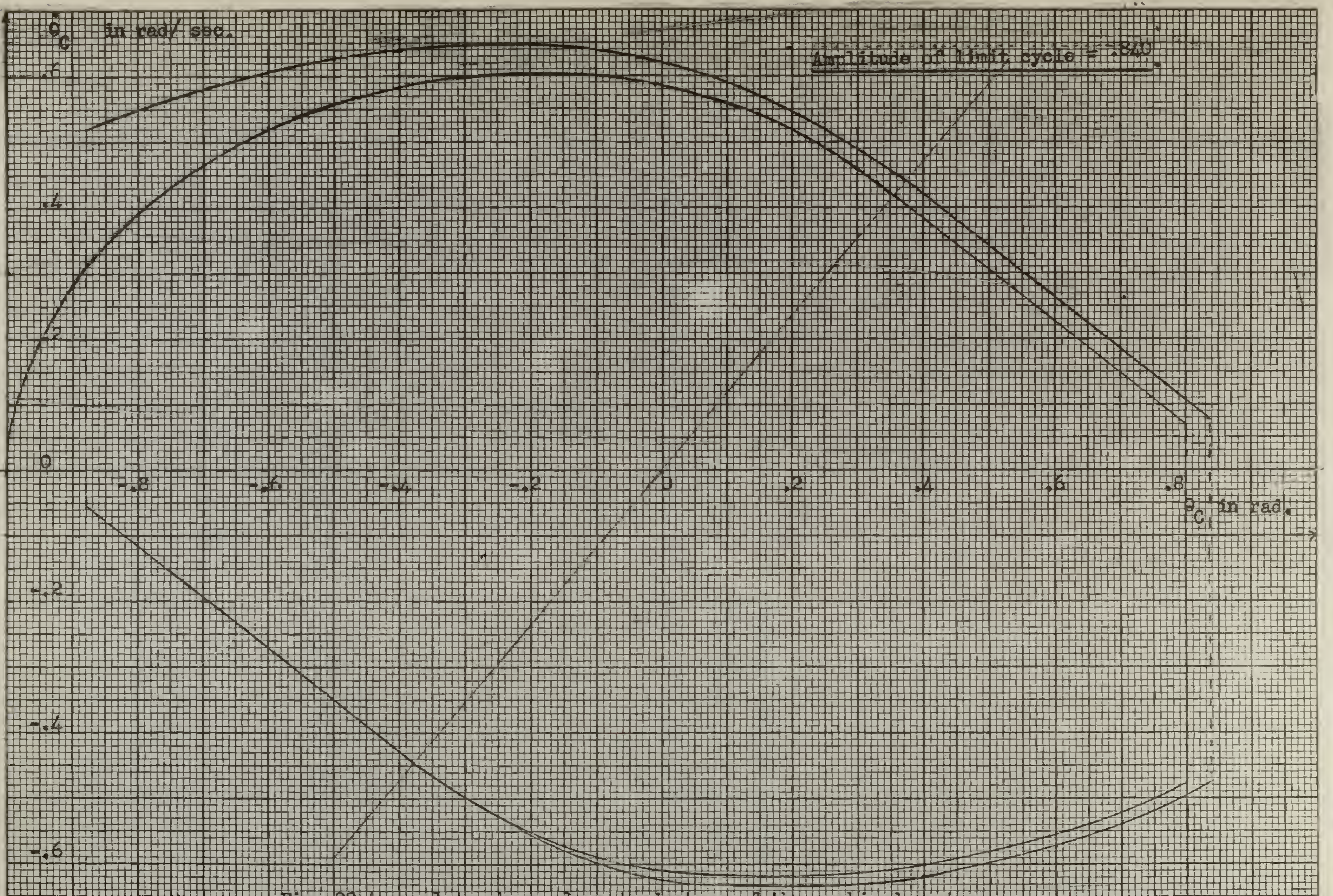

Fig: 28 A complete phase plane trajectory of the combined system

with Gain constant = 1.0, Unity step input and Backlash = 1.0

4. Analog computer technique.

For the combined system:

$$-\dddot{\theta}_C = -\left[\theta_R - 1.8\ddot{\theta}_C - \dot{\theta}_C - \theta_C\right] \tag{59}$$

This is an equation of a summer with 4 inputs i.e.

θ_R, $\ddot{\theta}_C$, $\dot{\theta}_C$ and θ_C.

$$\ddot{\theta}_C = -\frac{1}{s}\left[-\dddot{\theta}_C\right] \tag{60}$$

$$-\dot{\theta}_C = -\frac{1}{s}\left[-\ddot{\theta}_C\right] \tag{61}$$

$$\theta_C = -\frac{1}{s}\left[-\dot{\theta}_C\right] \tag{62}$$

Equations (60), (61) and (62) are equations for an integrator. Hence for the combined system we need 1 summer, 3 integrators and in addition 3 sign changers.

Summary for the combined system.

The output of amplifier 1 which in our case is θ_C is fed forward into amplifier 2 (an integrator). One output of this amplifier is fed back into amplifier 1 through a sign changer amplifier 6. The other output is fed forward into another integrator i.e. amplifier 3 Output #1 of amplifier 3 is fed back into amplifier 1. Output #2 is fed forward into integrator amplifier 4. The output of this is fed back through a sign changer into amplifier 1. The output of amplifier 2 and amplifier 4 are recorded into a X - Y plotter i.e. the output of amplifier 2 i.e. $\ddot{\theta}_C$ is fed into the Y - axis and the output of amplifier 4 i.e. θ_C is fed into the X - axis of the X-Y plotter. So we have a recording of acceleration versus displacement of the combined system.

The other recording is output of amplifier 7, which is $\dot{\theta}_C$ is fed into the Y - axis and the output of amplifier 4, which is θ_C is fed into the X - axis of the X - Y plotter. From this recording we have the phase - plane trajectory of velocity versus displacement of the combined system

The circuit diagram of the analog computer set-up of the combined system is shown in Fig. 30.

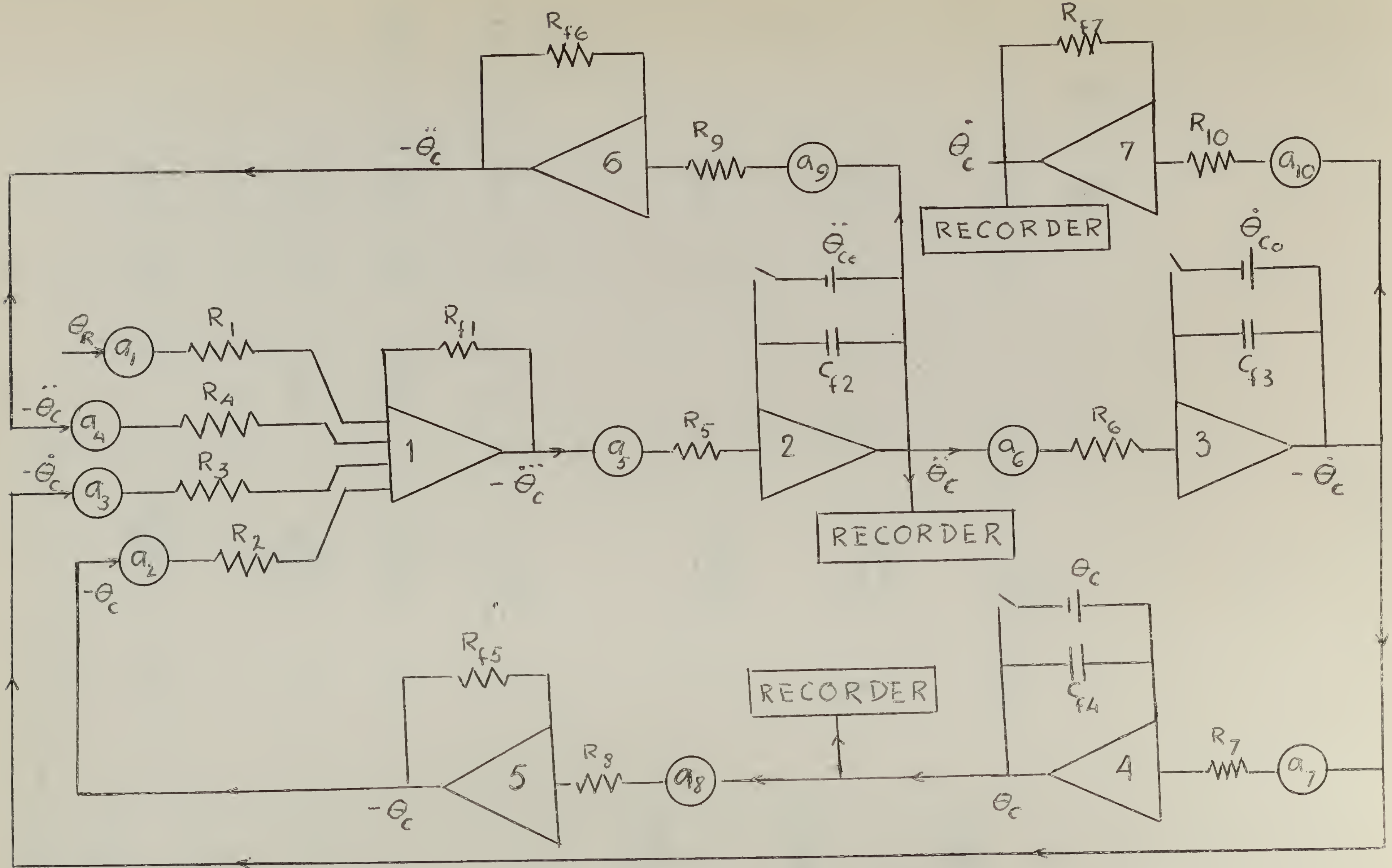

Fig. 30. A circuit diagram of computer set up # 1 for the combined system.

Calculation for the computer set up # 1.

Amplifier 1.

$$W_1 = \frac{a_1 \cdot R_{f1}}{R_1} = 1.0 \qquad a_1 = 1.0$$

$$W_2 = \frac{a_2 \cdot R_{f1}}{R_2} = 1.0 \qquad a_2 = 1.0$$

$$W_3 = \frac{a_3 \cdot R_{f1}}{R_3} = 1.0 \qquad a_3 = 1.0 \times .3 = .300$$

$$W_4 = \frac{a_4 \cdot R_{f1}}{R_4} = 1.8 \qquad a_4 = 1.8 \times .3 = .540$$

Amplifier 2.

$$W_5 = \frac{a_5}{C_{f2} \cdot R_5} = 1.0 \qquad a_5 = 1.0$$

Amplifier 3.

$$W_6 = \frac{a_6}{C_{f3} \cdot R_6} = 1.0 \qquad a_6 = 1.0$$

Amplifier 4.

$$W_7 = \frac{a_7}{C_{f4} \cdot R_7} = 1.0 \qquad a_7 = 1.0$$

Amplifier 5.

$$W_8 = \frac{a_8 \cdot R_{f5}}{R_8} = 1.0 \qquad a_8 = 1.0$$

Amplifier 6.

$$W_9 = \frac{a_9 \cdot R_{f6}}{R_9} = 1.0 \qquad a_9 = 1.0$$

Amplifier 7.

$$W_{10} = \frac{a_{10} \cdot R_{f7}}{R_{10}} = 1.0 \qquad a_{10} = 1.0$$

The components used in this computer set up # 1 are:

R_1 = 1.0 M-Ohms	R_{f1} = 1.0 Meg.
R_2 = 1.0 Meg	
R_3 = .3 Meg	
R_4 = .4 Meg	
R_5 = 1.0 Meg	C_{f2} = 1.0 microfarad
R_6 = 1.0 Meg	C_{f3} = 1.0 microfarad
R_7 = 1.0 Meg	C_{f4} = 1.0 microfarad
R_8 = 1.0 Meg	R_{f5} = 1.0 Meg
R_9 = 1.0 Meg	R_{f6} = 1.0 Meg
R_{10} = 1.0 Meg	R_{f7} = 1.0 Meg

The time scaling 1.0 sec.

The magnitude scaling for the displacement: 50 Volts = 1.0 rad.

The magnitude scaling for the velocity : 50 Volts = 1.0 rad/sec.

The magnitude scaling for the acceleration: 50 Volts = 1.0 rad/sec^2.

When the system separate, the equation of the load becomes:

$$\ddot{\theta}_L = -.8\dot{\theta}_L \tag{63}$$

$$\ddot{\theta}_L = \frac{1}{s}\left[\dot{\theta}_L\right] \tag{64}$$

$$\dot{\theta}_L = \frac{1}{s}\left[\theta_L\right] \tag{65}$$

The equation of the motor after separation becomes:

$$\dddot{\theta}_M = \left[.547\,\theta_R - .547\,\theta_L - 1.80\,\ddot{\theta}_M - .8\,\dot{\theta}_M\right] \tag{66}$$

$$\dddot{\theta}_M = \frac{1}{s}\left[\ddot{\theta}_M\right] \tag{67}$$

$$\ddot{\theta}_M = \frac{1}{s}\left[\dot{\theta}_M\right] \tag{68}$$

$$\dot{\theta}_M = \frac{1}{s}\left[\theta_M\right] \tag{69}$$

$$T_M = -\left[-.6\ddot{\theta}_M - .48\,\dot{\theta}_M\right] \tag{70}$$

Equation (64) and (65) are the equations for integrators. For the load equation we need two integrators and one sign changer. For the motor we need 3 integrators, one summer and 2 sign changers. The summery of the computer set up # 2 is as follows: One of the output of amplifier 1, which is $\dot{\theta}_L$ is fed back into amplifier 1. The other one is fed forward into amplifier 2. This amplifier has an output of θ_L (load displacement). The output of amplifier 3 is fed into the Y-axis of a X-Y plotter, while the X- axis gets its input from amplifier 7. So we have a recording of load displacement vs. motor displacement.

Amplifier 4 has 4 inputs i.e. θ_R, θ_L, $\ddot{\theta}_M$ and $\dot{\theta}_M$. One of the output of amplifier 4 is fed back into amplifier 4. The other one is fed forward into amplifier 5. There are two outputs of amplifier 5, one is fed back into amplifier 4 through a sign changer amplifier 8. The other one is fed forward into amplifier 6. Amplifier 9 has 2 inputs i.e. θ_M and $\ddot{\theta}_M$.

Recording points are:

$\dot{\theta}_M$ from amplifier 5 is fed into the Y - axis of a X - Y plotter.

θ_M from amplifier 7 is fed into the X - axis of the X - Y plotter.

Hence we have a phase plane trajectory of motor velocity versus motor displacement.

Motor torque T_M from amplifier 9 is fed into the Y - axis and the motor displacement from amplifier 7 is fed into the X - axis of the X - Y plotter.

From this we have the phase plane trajectory of motor torque versus motor displacement. The circuit diagram of the analog computer set up for the load and the motor after separation is shown in Fig. 31.

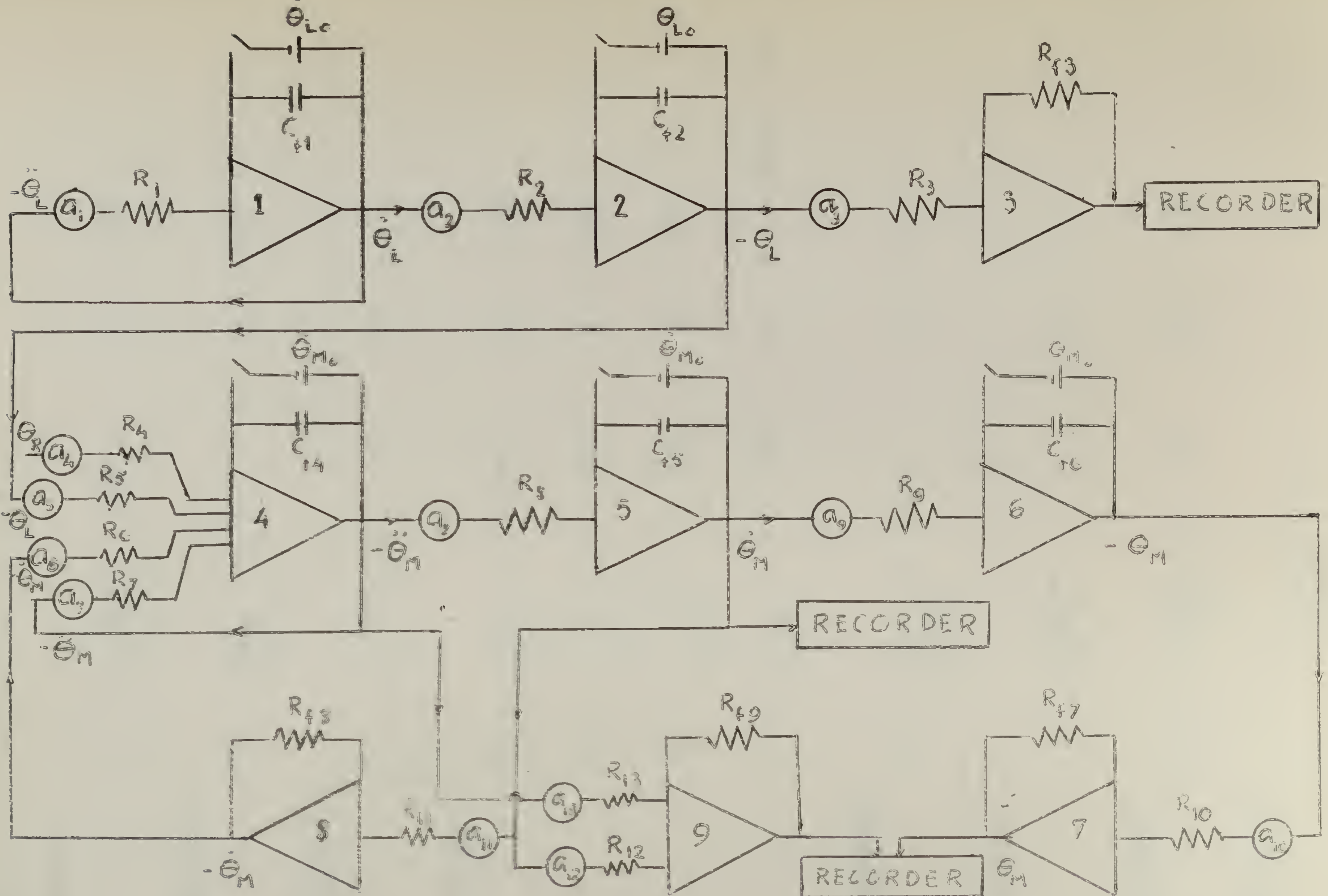

Fig: 31 a circuit diagram of computer set up # 2 for the load and motor after separation.

Calculation for the computer set up # 2.

<u>Amplifier 1.</u>

$$W_1 = \frac{a_1}{C_{f1} \cdot R_1} = .800 \qquad a_1 = .800$$

<u>Amplifier 2.</u>

$$W_2 = \frac{a_2}{C_{f2} \cdot R_2} = 1.0 \qquad a_2 = 1.0$$

<u>Amplifier 3.</u>

$$W_3 = \frac{a_3 \cdot R_{f3}}{R_3} = 1.0 \qquad a_3 = 1.0$$

<u>Amplifier 4.</u>

$$W_4 = \frac{a_4}{C_{f4} \cdot R_4} = .547 \qquad a_4 = .547$$

$$W_5 = \frac{a_5}{C_{f4} \cdot R_5} = .547 \qquad a_5 = .547$$

$$W_6 = \frac{a_6}{C_{f4} \cdot R_6} = .800 \qquad a_6 = .800$$

$$W_7 = \frac{a_7}{C_{f4} \cdot R_7} = 1.800 \qquad a_7 = 1.8 \times .5005 = .901$$

<u>Amplifier 5.</u>

$$W_8 = \frac{a_8}{C_{f5} \cdot R_8} = 1.0 \qquad a_8 = 1.0$$

<u>Amplifier 6.</u>

$$W_9 = \frac{a_9}{C_{f6} \cdot R_9} = 1.0 \qquad a_9 = 1.0$$

<u>Amplifier 7.</u>

$$W_{10} = \frac{a_{10} \cdot R_{f7}}{R_{10}} = 1.0 \qquad a_{10} = 1.0$$

<u>Amplifier 8.</u>

$$W_{11} = \frac{a_{11} \cdot R_{f8}}{R_{11}} = 1.0 \qquad a_{11} = 1.0$$

Amplifier 9.

$$W_{12} = \frac{a_{12} \cdot R_{f9}}{R_{12}} = .480 \qquad a_{12} = .480$$

$$W_{13} = \frac{a_{13} \cdot R_{f9}}{R_{13}} = .600 \qquad a_{13} = .600$$

The component used in the computer set up # 2 are:

R_1	= 1.0 Meg	C_{f1}	= 1.0 microfarad
R_2	= 1.0 Meg	C_{f2}	= 1.0 microfarad
R_3	= 1.0 Meg	R_{f3}	= 1.0 Meg
R_4	= 1.0 Meg	C_{f4}	= 1.0 microfarad
R_5	= 1.0 Meg		
R_6	= 1.0 Meg		
R_7	= .5005 Meg		
R_8	= 1.0 Meg	C_{f5}	= 1.0 microfarad
R_9	= 1.0 Meg	C_{f6}	= 1.0 microfarad
R_{10}	= 1.0 Meg	R_{f7}	= 1.0 Meg
R_{11}	= 1.0 Meg	R_{f8}	= 1.0 Meg
R_{12}	= 1.0 Meg	R_{f9}	= 1.0 Meg
R_{13}	= 1.0 Meg		

The time scaling and magnitude scaling used in the computer set up # 2 are the same as used in the previous computer set up.

5. The result of investigation.

We run experiments with different values of the system gain i.e.

a) $K = .95$

b) $K = 1.00$

c) $K = 1.20$

The result is plotted in Fig. 32. This figure shows the graph of the amplitude of limit cycle versus the backlash with different values of K. The graph tells us, that when we increase the amount of backlash, the amplitude of the limit cycle will increase too, provided we keep the gain constant. Now if we increase the gain, letting the amount of backlash constant, the amplitude of the limit cycle will increase. Notice that by decreasing the amount of the gain by .05 (i.e. from $K = 1.0$ to $K = .95$), the curve moves more to the right as does it to the left if we increase the system gain from $K = 1.0$ to $K = 1.20$. The values of the limit cycle with different gain constant is tabulated on table 8.

TABLE 8

Backlash in rad.	K - .95	Amplitude of limit cycle in rad. K = 1.00	K = 1.20
1.0	.598	.840	--
.9	.540	.810	--
.8	.425	.760	--
.7	.305	.725	.970
.6	--	.5625	.900
.5	--	.435	.790
.4	--	.355	.671
.3	--	--	.552
.2	--	--	--
.1	--	--	--

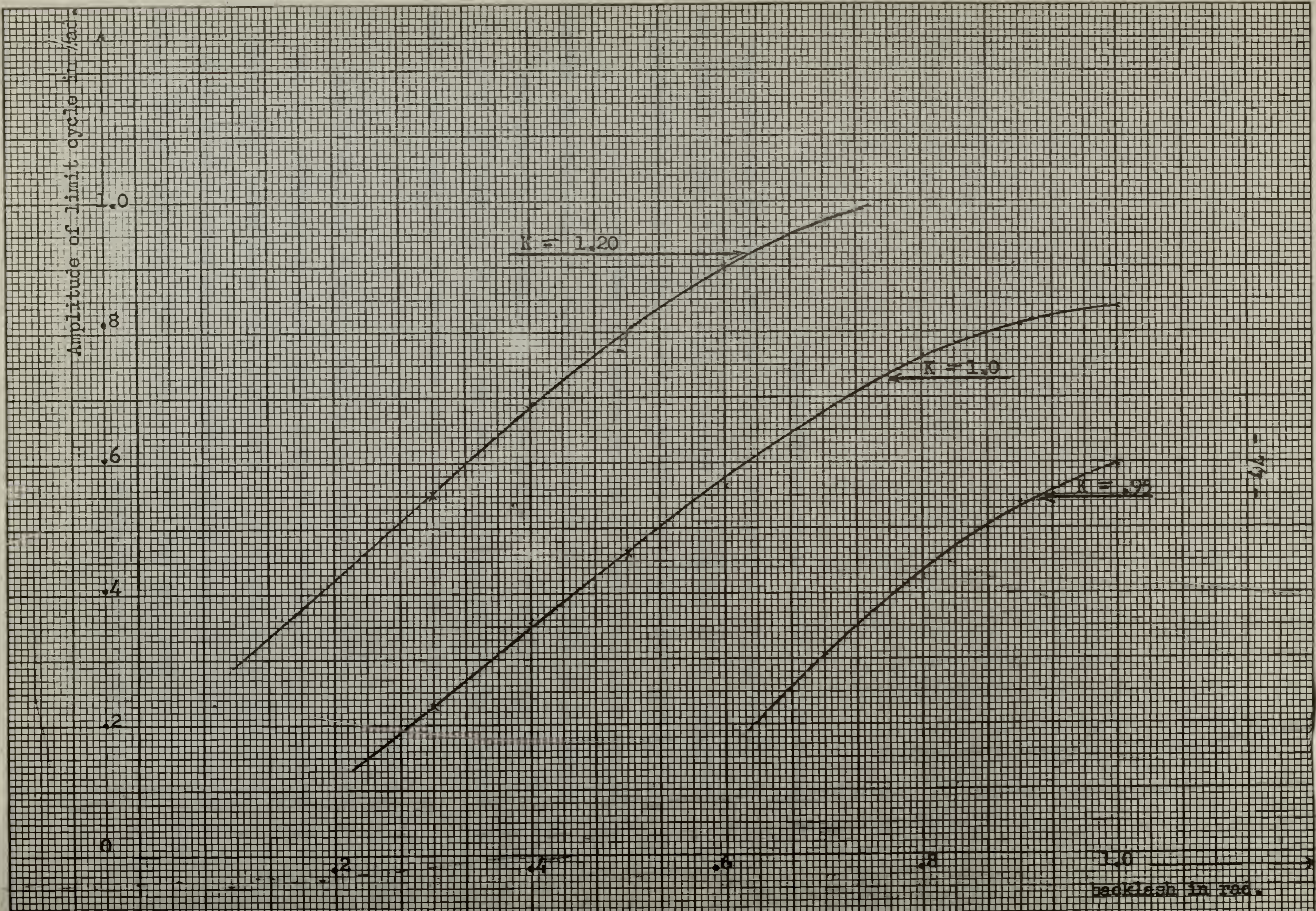

Fig. 32 The plot of amplitude of limit cycle vs backlash with several value of K.

5. Discussion and Conclusion

It is seen from the results of this study that backlash in a third order system may cause a limit cycle or may not cause a limit cycle depending on the amount of backlash and the system gain. With large amounts of backlash and low gain a limit cycle is unlikely. As the gain is increased it becomes more probable that a limit cycle will exist.

If a limit cycle does exist the amplitude of the limit cycle depends on both the amount of backlash and the gain. At low gain the amplitude of the limit cycle (in radians) may be considerably less than the amount of backlash (in radians). Conversely, at high gain the amplitude of the limit cycle may exceed the amount of backlash.

These conditions have been determined from study of a system with adjustable gain, but without variation of the inertia and friction, nor of the amounts of inertia and friction on each side of the backlash. These factors need further investigation.